PHYSICAL REVIEW A **80**, 011602(R) (2009)

Rotational fluxons of Bose-Einstein condensates in coplanar double-ring traps

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(Received 16 December 2008; published 13 July 2009)

Rotational analogs to magnetic fluxons in conventional Josephson junctions are predicted to emerge *in the ground state* of rotating tunnel-coupled annular Bose-Einstein condensates (BECs). Such topological condensate-phase structures can be manipulated by external potentials. We determine conditions for observing macroscopic quantum tunneling of a fluxon. Rotational fluxons in double-ring BECs can be created, manipulated, and controlled by external potentials in different ways than is possible in the solid-state system, thus rendering them a promising candidate system for studying and utilizing quantum properties of collective many-particle degrees of freedom.

DOI: 10.1103/PhysRevA.80.011602 PACS number(s): 03.75.Lm, 67.85.−d

I. INTRODUCTION

Remarkable experimental advances have made it possible to engineer cold atom systems to represent landmark models from completely different fields of physics. Examples include quantum phase transitions [1] and the Josephson effect [2]. Besides intriguing nonlinear dynamics, the Josephson effect shows macroscopic quantum phenomena with exciting prospects for applications [3]. Long Josephson junctions were used, e.g., to trap and study magnetic flux quanta, and the macroscopic quantum tunneling of such fluxons was observed [4]. Here we predict the existence of topological condensate-phase structure equivalent to superconducting fluxons in rotating Bose-Einstein condensates (BECs) that are confined in two concentric ring-shaped traps. The BECs in the individual rings are coupled by tunneling through a potential barrier at all azimuthal angles. The rotational fluxons can be understood as vortices that have entered the tunnel barrier. They show intriguing dynamical behavior and macroscopic quantum properties. Easier accessibility and more straightforward means of manipulation than possible in conventional Josephson junctions make rotational fluxons in tunnel-coupled BECs attractive for investigating fundamental problems ranging from models for cosmological evolution [5] (as recently studied in disk-shaped BECs [6]) to possibilities for realizing quantum information processing [7].

Recent successful efforts to create annular trapping geometries [8,9] and the routine use of trap rotation to simulate the effect a magnetic field has on charged particles [10] have motivated our present theoretical work. Unlike in the previously considered cases of vertically separated double-ring traps [11] or coupled elongated BECs [12,13], we predict rotational fluxons to occur in the ground states of the proposed system. In contrast to unconfined vortices in harmonically trapped or two- or three-dimensional annular [14] BECs, rotational fluxons are confined to the tunnel barrier region between the coupled rings for energetic reasons and thus take on properties of *topological solitons* [15]. Preparing ground-state solitons by cooling opens unprecedented opportunities for precision experiments on classical and quantum soliton dynamics. The phase structures are analogous to magnetic flux quanta occurring in a superconducting Josephson junction in a parallel magnetic field [3]. In coplanar double-ring BECs of mean radius R that rotate at frequency Ω, such fluxons appear due to a competition between trap rotation and coherent tunneling. While the former demands a tangential velocity ΩR different for the two rings, the latter favors identical tangential velocities. Spatially nonuniform or time-dependent potential differences between the two rings as caused by gradient or curved magnetic, gravitational, or optical fields generate forces on the fluxons. We also consider the conditions under which quantum tunneling of fluxons may be observed. Further theoretical and experimental studies of rotational fluxons in BECs promise to shed light on the behavior of collective excitations in interacting quasi-one-dimensional systems [16] and their macroscopic quantum properties [4].

Below we start by presenting the theoretical description of a tunnel-coupled coplanar double-ring system, which is based on the Gross-Pitaevskii (mean-field) equation with a radial double-well potential. Its solution provides the ground-state phase diagram as a function of trap-rotation frequency and tunnel-coupling strength as shown in Fig. 1. For finite tunnel coupling and a slow rotation, the phase difference between the two partial condensates in the individual rings vanishes. However, beyond a critical value of the rotation frequency, a quantized relative-phase winding between the two rings is accommodated, corresponding to a single rotational fluxon whose phase and density profiles are illustrated in Fig. 2. The phase structure can be detected experimentally by interferometry. For example after switching off the double-ring trap, both BECs will overlap in expansion and interfere destructively (resulting in a density node) at the azimuthal position of the fluxon. At higher critical values of rotation frequency, the number of rotational fluxons successively increases and a one-dimensional fluxon lattice is formed. After explaining our microscopic model and discussing numerical results, we present results from an effective

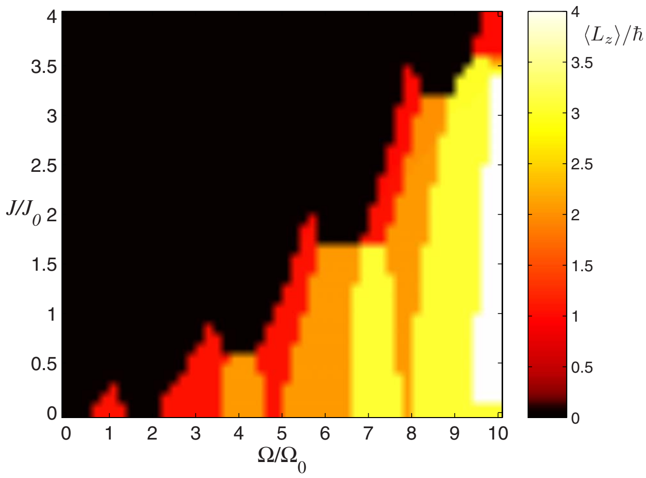

FIG. 1. (Color online) Phase diagram of coplanar double-ring BECs. We plot the difference of angular-momentum expectation values for condensate atoms in the outer and inner rings as a function of rotation frequency Ω and tunnel coupling J. Finite integer values observed at higher frequencies are associated with the presence of rotational fluxons. Results shown are obtained for a typical double-ring geometry with $d=0.36$ and $g/J_0=100$. Parameters and units are defined in the text.

hydrodynamic theory for BEC phase and density variables that captures the numerically observed behavior. A classical equation of motion for rotational fluxons can be derived, showing that time-dependent and/or spatially nonuniform potentials accelerate fluxons. We find an expression for the fluxon's inertial mass and discuss the possibility of quantum effects exhibited by these collective degrees of freedom.

II. THEORETICAL DESCRIPTION OF COPLANAR DOUBLE-RING BECS

We consider a situation where a BEC of atoms having mass M is confined by an external magnetic and/or optical trapping potential to two concentric rings with, in general,

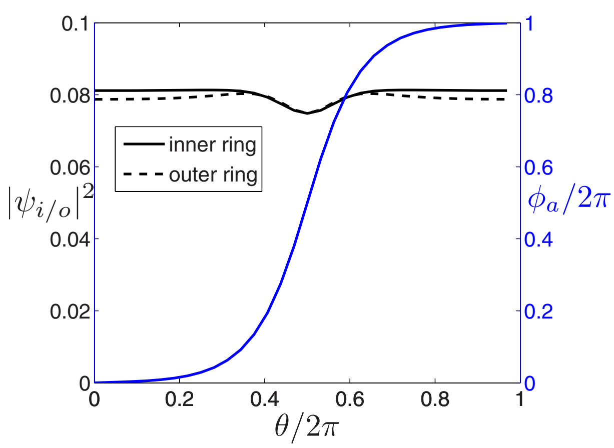

FIG. 2. (Color online) Single fluxon in the ground state of a double-ring BEC signified by the step-wise spatial variation in the relative phase ϕ_a for condensate fractions in the two rings. Also shown are partial condensate densities $|\psi_{i/o}|^2$ for the inner/outer ring. (θ is the azimuth. Parameters are $\Omega=5.8\Omega_0$, $J=1.9J_0$, and those used in Fig. 1.)

different radii R_i and R_o for the inner and outer rings, respectively. We assume that transverse excitations in the individual rings are frozen out, allowing for a purely one-dimensional description. In addition, the two rings are coupled linearly by tunneling through a barrier with an associated tunnel energy J. Following Ref. [11], we consider the coupled Gross-Pitaevskii equations for the inner and outer ring wave functions $\psi_i(\theta,t)$ and $\psi_o(\theta,t)$, respectively, which are given by

$$i\hbar\partial_t\psi_{i/o} = \left[-\frac{\hbar^2}{2MR_{i/o}^2}\partial_\theta^2 + i\hbar\Omega\partial_\theta + \beta \mp \delta + g_{i/o}|\psi_{i/o}|^2\right]\psi_{i/o} - J\psi_{o/i}. \tag{1}$$

Here $\delta=(E_o-E_i)/2$ and $\beta=(E_o+E_i)/2$ in terms of the single-well bound-state energies $E_{i/o}$. Rotation around the trap axis with frequency Ω is imposed by any (initial) anisotropy in the trapping potential. Using the normalization condition $\Sigma_{\alpha=i,o}\int|\psi_\alpha|^2 d\theta=1$ the nonlinear coupling energies are $g_{i/o}=ng_{1D}^{(i/o)}$, where $n=N/(2\pi R)$ is an average linear particle density and $g_{1D}^{(i/o)}$ is the effective one-dimensional coupling strength [17]. For convenience, we introduce the effective trap radius $R=\sqrt{2}R_oR_i/\sqrt{R_o^2+R_i^2}$ and $d=(R_o^2-R_i^2)/(R_o^2+R_i^2)$, which is a measure of the radial wells' separation, as parameters instead of $R_{i/o}$. We have solved Eq. (1) using a Fast-Fourier-Transformation (FFT)-based pseudospectral method with imaginary-time propagation [18] to find the ground states of double-ring BECs. For simplicity, we assumed $g_i=g_o\equiv g$. To compensate a trivial energy shift between states in the inner and outer wells due to finite rotation, we have set δ to $\delta_*\equiv M\Omega^2R^2d/[2(1-d^2)]$ for Figs. 1 and 2.

In the absence of interactions (i.e., $g=0$), stationary solutions of Eq. (1) can be labeled by the quantum number $\hbar m$ of the angular-momentum component $L_z\equiv -i\hbar\partial_\theta$ along the symmetry axis of the trap. The condensate wave functions in the inner and outer rings will be given by $\psi_{i/o}(\theta)\propto e^{im\theta}$, and the phase difference between condensate amplitudes in the two rings will vanish at every point θ. However, a finite g introduces a mixing of amplitudes with different m values in the condensate wave function, enabling the appearance of nontrivial structure in the relative phase. To illustrate this point quantitatively, we calculated the difference of expectation values of L_z per particle in the outer- and inner-ring condensate fractions, i.e., $\langle\Delta L_z\rangle\equiv\langle L_z\rangle_o-\langle L_z\rangle_i$, where $\langle L_z\rangle_{i/o}=\langle\psi_{i/o}|L_z|\psi_{i/o}\rangle/\langle\psi_{i/o}|\psi_{i/o}\rangle$. In Fig. 1, $\langle\Delta L_z\rangle/\hbar$ is plotted as a function of tunnel coupling J [measured in units of $J_0=\hbar^2/(2MR^2)$] and rotation frequency Ω [measured in units of $\Omega_0=\hbar/(2MR^2)$] for a particular double-ring geometry. Regions with finite *integer* $\langle\Delta L_z\rangle/\hbar$ are observed, which correspond to ground states with (one or more) rotational fluxons present. A representative example for such a fluxon's relative-phase and partial-condensate density profiles is shown in Fig. 2.

Basic features of the phase diagram shown in Fig. 1 can be understood by a variational consideration that assumes (i) strong nonlinear coupling g such that both rings are populated with equal density and (ii) the condensate wave function in each ring to be given by an L_z eigenstate, $\psi_{i/o}^{(var)}(\theta)=e^{im_{i/o}\theta}/\sqrt{4\pi}$. The values of $m_{i/o}$ are determined by a competition between tunneling, which tends to enforce equal phase for condensate fractions in both the inner and outer rings ($m_\mathrm{i}=m_\mathrm{o}\equiv m_*$) and rotation. The latter favors the two condensate fractions to have, in general, different angular momenta determined by the rotation frequency and the ring radii ($m_\mathrm{i/o}=\widetilde{m}_\mathrm{i/o}\equiv \mathrm{Int}[MR_\mathrm{i/o}^2\Omega/\hbar+1/2]$). It is straightforward to derive the energy functional of the system,

$$\mathcal{E}[m_\mathrm{o},m_\mathrm{i}]=\frac{\hbar^2}{4M}\left(\frac{m_\mathrm{o}^2}{R_\mathrm{o}^2}+\frac{m_\mathrm{i}^2}{R_\mathrm{i}^2}\right)-\frac{\hbar\Omega}{2}(m_\mathrm{o}+m_\mathrm{i})-J\delta_{m_\mathrm{o},m_\mathrm{i}}. \tag{2}$$

The condition $\mathcal{E}[\widetilde{m}_\mathrm{o},\widetilde{m}_\mathrm{i}]=\mathcal{E}[m_*,m_*]$ defines a critical value $J_\mathrm{cr}\equiv M\Omega^2R^2d^2/[2(1-d^2)]$. For $J>J_\mathrm{cr}$, the state having $m_\mathrm{i}=m_\mathrm{o}\equiv m_*$ would be expected to be the ground state, corresponding to the black region in Fig. 1. In the opposite case, the phase gradient for partial-condensate wave functions in the two rings will be different, essentially realizing a vortex (or several vortices) in the phase difference between the two rings. Such a situation is signified by the brighter colored regions in Fig. 1. The variational estimate of J_cr yields a reasonably accurate description of the actual phase boundaries seen in the numerically obtained phase diagram.

III. EFFECTIVE ANALYTICAL THEORY OF FLUXON PHASE PROFILE AND DYNAMICS

To obtain a more detailed understanding of fluxons in coupled annular BECs, we consider the dynamics of their collective phase and density variables. This approach applies equally well to coplanar and vertically separated double-ring traps. Writing the partial-condensate wave functions as $\psi_\mathrm{i/o}=|\psi_\mathrm{i/o}|\exp\{i\phi_\mathrm{i/o}\}$, we define symmetric and antisymmetric combinations of their modulus and phase and express the Lagrangian of the double-ring system in terms of these new quantities. It is possible to derive a closed equation of motion for the phase difference $\phi_\mathrm{a}=\phi_\mathrm{o}-\phi_\mathrm{i}$ that is accurate to first order in the typically small quantity E_R/g, where $E_R=\hbar^2/(2MR^2)$ is the scale of energy quantization on the ring. Its lengthy analytical expression is omitted.

In the stationary limit and to leading (zeroth) order in E_R/g, we find

$$(1-d^2)E_R\partial_\theta^2\phi_\mathrm{a}-2J\sin\phi_\mathrm{a}=0, \tag{3}$$

which has a single-soliton (i.e., fluxon) solution [19]

$$\phi_\mathrm{a}^{(\mathrm{fl})}(\theta,\theta_0)=\pi+2\arcsin\left[\mathrm{sn}\left(\frac{\kappa(\theta-\theta_0)}{k}\middle| k\right)\right]. \tag{4}$$

Here $\mathrm{sn}(u|k)$ is a Jacobi elliptic function [20] whose parameter k is determined from the transcendental relation

$$\pi\kappa=kK(k), \tag{5}$$

involving the complete elliptic integral of the first kind, and $\kappa=\sqrt{2J/[(1-d^2)E_R]}$. Hence, fluxons emerge as stationary phase configurations, as seen in our numerical calculations. The dimensionless parameter $\kappa\equiv R/(\sqrt{1-d^2}\lambda_J)$ can be interpreted as the ratio of the quadratic mean radius of the trap $R/\sqrt{1-d^2}=\sqrt{(R_o^2+R_i^2)/2}$ and the physical length scale of the fluxon $\lambda_J=\hbar/(2\sqrt{MJ})$, which is set by the tunnel coupling.

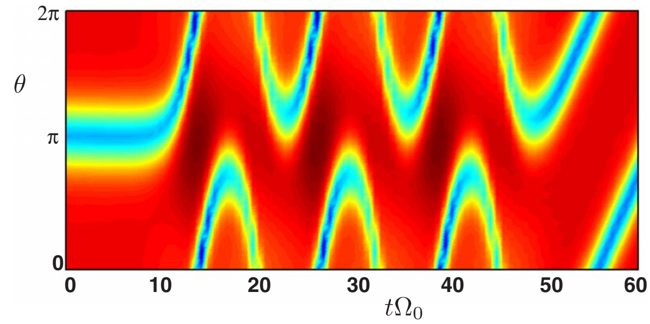

FIG. 3. (Color online) Dynamics of a single fluxon under a time-dependent external field according to Eq. (1) (we show the total density $|\psi_i|^2+|\psi_o|^2$). An external potential gradient across the double-ring trap corresponding to $\delta=0.2\beta=0.2J_0\cos(\theta)$ is smoothly turned on at $t=10\Omega_0^{-1}$ and off again at $t=50\Omega_0^{-1}$. Other parameters are $J=J_0$ and $g=100J_0$.

To obtain a dynamical equation for a slowly moving fluxon, we insert the ansatz $\phi_\mathrm{a}(\theta,t)=\phi_\mathrm{a}^{(\mathrm{fl})}(\theta,\theta_0(t))$ into the equation of motion for the phase difference. Here $\theta_0(t)$ is the instantaneous position of the fluxon. Straightforward algebraic manipulation yields a Newton-like equation of motion:

$$M_\mathrm{fl}\ddot{\theta}_0R=F_\mathrm{fl}. \tag{6}$$

The fluxon's dynamical mass is $M_\mathrm{fl}=2\sqrt{E_R/g}\mathcal{I}_\mathrm{fl}M$ with the dimensionless moment of inertia given by

$$\mathcal{I}_\mathrm{fl}=(1+d^2)\int_0^{2\pi}\frac{d\theta}{4\pi}[\partial_\theta\phi_\mathrm{a}^{(\mathrm{fl})}]^2, \tag{7a}$$

$$\equiv\frac{1+d^2}{\pi}\frac{\kappa}{k}E\left(\frac{2\pi\kappa}{k}\middle| k\right), \tag{7b}$$

where $E(u|k)$ is the incomplete elliptic integral of the second kind [20]. The general expression for the force (torque) on the fluxon is [21]

$$F_\mathrm{fl}=\sqrt{\frac{2M}{g}}\int_0^{2\pi}\frac{d\theta}{2\pi}\partial_\theta\phi_\mathrm{a}^{(\mathrm{fl})}\times\left\{\partial_t\delta+d\partial_t\beta+d\left(\frac{(1-d^2)E_R}{2\hbar}\partial_\theta\phi_\mathrm{a}^{(\mathrm{fl})}-\Omega d\right)\partial_\theta\delta\right\}. \tag{8}$$

Equations (5) and (7b) define a universal relationship between the fluxon's dimensionless moment of inertia $\mathcal{I}_\mathrm{fl}$ and the variable κ. The limiting value of $\mathcal{I}_\mathrm{fl}$ for small trap size ($\kappa\ll 1$) is a constant $[(1+d^2)/2]$, whereas a linear dependence $[2(1+d^2)\kappa/\pi]$ is realized for large ring traps ($\kappa\gg 1$).

Inspection of Eq. (8) reveals that fluxons subject to spatially nonuniform δ and/or time-dependent δ or β will experience a force. This feature is confirmed by our numerical solution of Eq. (1), an example being shown in Fig. 3. In the case of spatially uniform $\delta(\theta,t)\equiv\delta_0(t)$ and $\beta=0$, the force simplifies to $\sqrt{2M/g}\sigma\dot{\delta}_0$, which is similar to the result found previously [13] for phase-imprinted fluxons in a junction between two parallel *linear* BECs. The sign $\sigma = \mathrm{sgn}[\phi_a^{(\mathrm{fl})}(2\pi) - \phi_a^{(\mathrm{fl})}(0)]$ is the topological charge of the fluxon related to its orientation. Here we found the expression for the force felt by fluxons in the more general case with $d \neq 0$.

If the external fields are time independent and the fluxon length λ_J is smaller than the length scale of spatial variations of δ, Eq. (8) can be integrated and written as $F_{\mathrm{fl}} = -R^{-1}\partial_\theta V$, where V is a potential energy. For $\kappa \gg 1$ and to leading order in d, we obtain

$$V(\theta) = \left(\sigma \frac{\hbar \Omega d^2}{\sqrt{E_R g}} - \frac{2\sqrt{2}d}{\pi}\sqrt{\frac{J}{g}}\right)\delta(\theta) \tag{9}$$

for the potential and $M_{\mathrm{fl}} = M\sqrt{32J/(\pi^2 g)}$ for the dynamical fluxon mass.

IV. MACROSCOPIC QUANTUM TUNNELING

Describing the effects of quantum and thermal fluctuations on the fluxon dynamics can proceed in analogy to the established treatment of Josephson vortices in superconducting junctions [3]. In particular, the possibility of fluxon (macroscopic quantum) tunneling can be included [22] by direct quantization of the classical equation of motion [Eq. (6)]. A rough estimate for tunneling of a fluxon through a potential barrier of height ΔV and length Δl from the WKB method yields the probability $P \approx \exp(-2\Delta l \sqrt{2M_{\mathrm{fl}}\Delta V}/\hbar)$. In order to have $P \gtrsim 1/e$ with $\Delta l \approx \lambda_J$, we need $\sqrt{Jg} \gtrsim \Delta V$. Assuming a double-ring configuration as proposed in Ref. [9] with $R \approx 50$ μm, it may be feasible to achieve $g/k_B \approx 2$ μK and $J/k_B \approx 0.05$ μK and observe quantum tunneling through barriers $\Delta V/k_B \lesssim 0.3$ μK at sufficiently low temperatures.

Quasiparticle excitations present at finite temperature will act as a damping mechanism for fluxon motion [3] and, at the same time, as a source of quantum decoherence (thus suppressing fluxon tunneling [22]).

V. DISCUSSION AND CONCLUSIONS

We have predicted fluxonlike topological structure in the relative phase of condensate fractions in the ground state of BECs in rotating double-ring traps. These rotational fluxons are accelerated by spatially varying external potentials that couple asymmetrically to the two rings and/or time-dependent potentials. Macroscopic quantum tunneling of fluxons may become observable and would serve the long sought goal of preparing macroscopic quantum superposition states of BECs (see, e.g., Ref. [23]). Future studies will focus on details of fluxons' quantum properties and possible applications [5,7].

ACKNOWLEDGMENTS

J.B. was supported by the Marsden Fund Council (Contract No. MAU0706) from Government funding administered by the Royal Society of New Zealand.

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