Sign of coupling in barrier-separated Bose-Einstein condensates and stability of double-ring systems

J. Brand
Centre for Theoretical Chemistry and Physics and Institute of Natural Sciences, Massey University (Albany Campus), Private Bag 102 904, North Shore MSC, Auckland 0745, New Zealand

T. J. Haigh*
Institute of Fundamental Sciences, Massey University (Manawatu Campus), Private Bag 11 222, Palmerston North 4442, New Zealand

U. Zülicke
Institute of Fundamental Sciences, Massey University (Manawatu Campus), Private Bag 11 222, Palmerston North 4442, New Zealand, Centre for Theoretical Chemistry and Physics, Massey University (Albany Campus), Private Bag 102 904, North Shore MSC, Auckland 0745, New Zealand
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We revisit recent claims about the instability of nonrotating tunnel coupled annular Bose-Einstein condensates leading to the emergence of angular momentum Josephson oscillation [Phys. Rev. Lett. **98**, 050401 (2007)]. It was predicted that all stationary states with uniform density become unstable in certain parameter regimes. By careful analysis, we arrive at a different conclusion. We show that there is a stable nonrotating and uniform ground state for any value of the tunnel coupling and repulsive interactions. The instability of an excited state with π phase difference between the condensates can be interpreted in terms of the familiar snake instability. We further discuss the sign of the tunnel coupling through a separating barrier, which carries significance for the nature of the stationary states. It is found to always be negative for physical reasons.

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I. INTRODUCTION

Bose-Einstein condensates (BECs) located in different minima of an external potential created by magnetic or light forces and coupled by tunneling through a potential barrier have been the host to many exciting developments and discoveries in recent years [1]. Phenomena explored include analogs of the Josephson effect in double or multiple quantum-well structures [2–4], gap solitons of repulsive BECs [5], and quantum phase transitions [6]. Often these systems are modeled by considering just one mode per potential minimum and their linear coupling provided by tunneling through a separating barrier. These simplified models, which are usually labeled as two-mode or multiple-mode models, variants of the Bose-Hubbard model, or the discrete nonlinear Schrödinger equation, are tailored to describe certain properties or aspects of the dynamics of the many-body system under investigation. Although there is an abundance of literature on such models [7–16], there still appear to exist misconceptions about the nature of the effective model parameters, especially the sign of the tunnel coupling, as only few authors attempt to calculate such parameters based on a more complete theoretical treatment [11,13,14]. In addition, classic papers disagree in explicit or implicit statements made about the sign of the tunneling constant [7,8,17]. It is the purpose of this Brief Report to clarify the matter.

The sign of the tunnel coupling bears special significance in systems where the tunneling appears over an extended (at least one-dimensional) region of space. Such systems have recently been analyzed by Bouchoule [18] and Kaurov and Kuklov [19], who studied two parallel tunnel-coupled cigar-shaped BECs. Related earlier studies [20] considered the propagation of BECs in adiabatically split parallel wave guides. In another recent work, Lesanovsky and von Klitzing investigated the stability of tunnel-coupled annular BECs [21]. The latter paper points to an interesting dynamical instability leading to the spontaneous formation of angular momentum fluctuations. We will show in the following that the sign on the tunnel coupling bears consequences on the nature and stability of the stationary states found in the mean-field treatment of tunnel-coupled BECs. Specifically, we find that the system studied by Lesanovsky and von Klitzing has a stable ground state for any value of the tunnel coupling and repulsive interactions. The instability of an *excited* state with π phase difference between the condensates can be interpreted in terms of the familiar snake instability [22]. The ground state of a rotating co-planar double-ring system is discussed in Ref. [23].

We examine the stationary states of double-ring BECs in Sec. II. A careful analysis of the sign of the tunnel coupling used in effective models for BECs in double-well traps follows in Sec. III. Conclusions are presented in Sec. IV.

II. STABILITY OF STATIONARY STATES IN DOUBLE-RING BECs

The calculation performed in Ref. [21] starts from a number of generally reasonable assumptions. Under the condition that radial excitations of the vertically stacked annular BECs are suppressed by the trapping potentials and the only mechanism for coupling the two systems is via tunneling through a potential barrier, the Gross-Pitaevskii equation for the

*Present address: ARC Centre of Excellence for Quantum-Atom Optics, School of Physical Sciences, University of Queensland, Brisbane, QLD 4072, Australia.

two-mode spinor wave function (χ_u, χ_d) specializes to

$$i\partial_\tau \chi_{u/d} = -\partial_\varphi^2 \chi_{u/d} - |\kappa|\,\chi_{d/u} + \gamma\,|\chi_{u/d}|^2 \chi_{u/d}. \qquad (1)$$

Here $\chi_{u(d)}$ is the condensate wave function for atoms in the upper (lower) ring. Our Eq. (1) agrees with Eq. (2) of Ref. [21], except that we explicitly indicate the negative sign of the tunnel coupling. We give detailed reasons for the relevance of the sign of the tunnel coupling in Sec. III, where we also show that the tunnel coupling is indeed negative. At this point, we only note that the tunnel coupling κ was assumed to be positive in Ref. [21] (see their Fig. 1), in contradiction to our findings.

The most general form of the polar-angle-dependent wave function can be written as a Fourier series, $\chi_{u/d} = (2\pi)^{-\frac{1}{2}} \sum_m \alpha_m^{(u/d)} e^{im\varphi}$. Inserting this ansatz into Eq. (1) and equating coefficients of the orthogonal Fourier components, we find

$$i\partial_\tau \alpha_m^{(u/d)} = m^2 \alpha_m^{(u/d)} - |\kappa|\,\alpha_m^{(d/u)} + \frac{\gamma}{2\pi}\sum_{n,n'} \alpha_n^{(u/d)} \alpha_{n'}^{*(u/d)} \alpha_{m-n+n'}^{(u/d)}. \qquad (2)$$

This result differs from the corresponding Eq. (3) in Ref. [21] in the tunnel coupling and the nonlinear term.

As a first approximation, it is reasonable to assume that only the $m = 0$ mode is occupied in each of the two annuli. Straightforward calculation yields the new ground and excited states of the coupled-annuli system, which are the symmetric and antisymmetric superpositions of single-well states having chemical potential $\mu_\pm = \varepsilon \mp |\kappa|$, respectively. $\varepsilon = \gamma N_0/(2\pi)$ is defined in terms of the equal number of atoms, N_0, in each well as in Ref. [21]. In order to study the stability of these states, finite but small amplitudes in the $m \neq 0$ modes are assumed:

$$\alpha_{m\neq 0}^{(u/d)} = e^{-i\mu_\pm \tau}\left[u_{m,\pm}^{(u/d)} e^{-i\omega\tau} + v_{m,\pm}^{*(u/d)} e^{i\omega\tau}\right]. \qquad (3)$$

Here the subscript $\pm$ distinguishes perturbations to the ground and excited states, respectively. Inserting the perturbation (3) into Eq. (2) and linearizing in the small amplitudes u, v yields

$$\begin{aligned} \omega u_{m,\pm}^{(u/d)} &= (m^2 + \varepsilon \pm |\kappa|)\, u_{m,\pm}^{(u/d)} + \varepsilon v_{-m,\pm}^{(u/d)} - |\kappa| u_{m,\pm}^{(d/u)}, \\ -\omega v_{-m,\pm}^{(u/d)} &= (m^2 + \varepsilon \pm |\kappa|)\, v_{-m,\pm}^{(u/d)} + \varepsilon u_{m,\pm}^{(u/d)} - |\kappa| v_{-m,\pm}^{(d/u)}. \end{aligned} \qquad (4)$$

The upper (lower) sign refers to the symmetric ground (antisymmetric excited) state. Crucial differences between our Eq. (4) and Eq. (5) in Ref. [21] result in markedly different excitation spectra. We find that both the symmetric (ground) state and antisymmetric (excited) state share one branch,

$$\omega_1 = \sqrt{(m^2 + \varepsilon)^2 - \varepsilon^2}\,, \qquad (5a)$$

whose frequency is independent of the tunnel coupling. This was also found in Ref. [21]. In contrast to the results of these authors, however, we find that the second branch differs for the two states:

$$\omega_{2,\pm} = \sqrt{(m^2 + \varepsilon \pm 2|\kappa|)^2 - \varepsilon^2}. \qquad (5b)$$

Clearly, $\omega_{2,+}$ is always real for repulsive BECs ($\varepsilon > 0$), implying stability of the symmetric (ground) state of the coupled annular condensates. In contrast, the antisymmetric (excited) state will become unstable for $\varepsilon > |\kappa| - m^2/2 > 0$, signified by $\omega_{2,-}$ becoming imaginary in this range. Our own numerical simulations of the time evolution of the antisymmetric state seeded with a small amount of noise show the development of angular momentum Josephson junctions similar to those shown in Figs. 2 and 3 of Ref. [21].

In attractive condensates where $\varepsilon < 0$, imaginary solutions of ω_1 for $2\varepsilon < -m^2$ indicate the well-known modulational instability toward the formation of localized peaks (bright solitons) in the individual rings. For the symmetric state, $\omega_{2,+}$ does not add new instabilities (with imaginary solutions for $\varepsilon < -m^2/2 - |\kappa|$). The antisymmetric state, however, is further destabilized by the tunnel coupling due to imaginary frequencies of $\omega_{2,-}$ at $\varepsilon < |\kappa| - m^2/2 < 0$.

In our analysis so far we have assumed that the sign of the coupling constant κ is negative. This leads to the symmetric state with $\alpha_0^{(d)} = \alpha_0^{(u)} = \text{constant} \cdot e^{i\mu_+\tau}$ and $\alpha_{m\neq 0}^{(u/d)} = 0$ with $\mu_+ = \varepsilon - |\kappa|$ being the ground state. Let us now briefly consider the consequences of the (hypothetical) case of a positive coupling constant $\kappa > 0$. The analysis of Sec II can be carried out the same way as before, with the difference that $|\kappa|$ should be replaced by $-|\kappa|$ in all formulas. It is easily seen that, in this case, the antisymmetric state with $\alpha_0^{(d)} = -\alpha_0^{(u)}$ will be the ground state. Since the sign change also affects Eq. (5b), we find the antisymmetric state being stable (for $\varepsilon > 0$) and the symmetric one becoming unstable. However, since the roles of these states have changed, we still find that the ground state is stable for repulsive BECs.

III. NATURE OF THE TUNNEL COUPLING

In order to determine the correct sign and value of the coupling constant κ appearing in Eq. (1), we briefly revisit the derivation of this model. Generally, two types of approaches have been followed for deriving effective two-mode models: (a) The mode functions and tunnel parameter are derived from solutions of the single-particle Schrödinger equation. This approach is commonly used when deriving the fully quantum mechanical Bose-Hubbard model [15,17] and was the basis of Refs. [7,21]. In this case both the sign and the value of κ are completely independent of particle number or interaction strength. In the context of the cylindrically symmetric double-ring system it is further possible to completely separate the azimuthal, radial, and axial degrees of freedom. As a consequence, the tunnel coupling κ is also independent of angular momentum along the trap axis (azimuthal excitations). (b) When the mode functions and model parameters are calculated from the nonlinear Schrödinger equation [8,13] or by a variational procedure within the interacting system [11,12], this separability of the spatial degrees of freedom is lost. While this procedure has the potential to reproduce the dynamics of systems with large particle number more accurately than the type (a) approaches, the tunnel coupling κ, in principle, becomes dependent on particle number and interaction strength as well as angular momentum due to centrifugal distortions. While exploring the consequences of these dependencies in detail goes beyond the scope of this Brief Report, we argue that the sign of κ is fixed by requiring that the low-energy physics is described correctly in a qualitative manner.

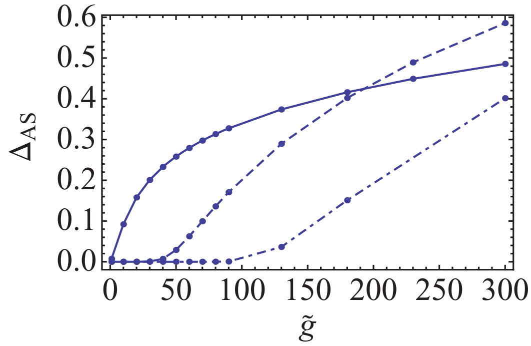

FIG. 1. (Color online) Energy difference $\Delta_{AS} = E_A - E_S$ between the lowest antisymmetric and symmetric eigenstates of a quadratic-plus-quartic double-well potential, plotted as a function of the dimensionless effective interaction strength $\tilde{g}$. The fact that $\Delta_{AS} \geqslant 0$ indicates that the symmetric (nodeless) state remains the ground state even in the limit where the atoms interact strongly. Double-well parameters [see Eq. (7)] are $\xi_0 = 5$ and $h = 0.002$ (solid curve), 0.02 (dashed curve), and 0.05 (dot-dashed curve).

For the purpose of determining the sign of κ, the azimuthal degree of freedom in the double-ring model of Ref. [21] is irrelevant. It suffices to consider the problem of a BEC in a one-dimensional (1D) double-well potential, as in Refs. [7,14]. Generalization to multiple wells and different geometries (coupled cigars or pancakes) are straightforward.

The goal of a two-mode model is generally to correctly describe the ground and low-lying excited states of the system. Initially we consider the linear Schrödinger equation of a particle in a symmetric double well as relevant to type (a) derivations. The quantity that is obtainable from the 1D model and carries unambiguous information about the sign of the tunnel coupling is the energy difference $\Delta_{AS} = E_A - E_S$ between the antisymmetric state with one node and the nodeless symmetric state. The node theorem of quantum mechanics [24] guarantees that the nodeless symmetric state in a 1D double-well potential must be the ground state; thus $\Delta_{AS} \geqslant 0$ and, consequently, the correct sign of κ is negative.

We now consider the case (b) of two-mode models designed to approximate the system in an interacting (nonlinear) regime. We decide to choose the parameters of the two-mode model in order to reproduce Δ_{AS} as found from a 1D Gross-Pitaevskii equation. The ordering of eigenvalues by the number of nodes in the wave function is now no longer guaranteed by the node theorem of linear quantum mechanics, and we are not aware of a nonlinear generalization of this theorem. However, we find by numerical calculation that the ordering is preserved under repulsive interactions. The main result of this section is the dependence of Δ_{AS} on the nonlinear interaction strength $\tilde{g}$, shown in Fig. 1. As can be seen from Fig. 1, the presence of a repulsive nonlinear interaction does not change the sign of Δ_{AS} and therefore κ remains negative. We now present details of our calculation.

Starting from the three-dimensional Gross-Pitaevksii equation for a BEC in a double-well or double-ring trap and employing a separation ansatz, an effective 1D equation describing the dynamics in the direction perpendicular to the potential barrier can be derived:

$$\frac{\mu}{\varepsilon_0}\phi(\xi) = \left[-\frac{d^2}{d\xi^2} + V_{\rm dw}(\xi) + \tilde{g}|\phi(\xi)|^2\right]\phi(\xi). \qquad (6)$$

Here the energy scale ε_0 and length scale a_0 defined by the trap are used as units for all energies and the spatial coordinate, respectively, and the condensate wave function ϕ is normalized to unity. We introduced the dimensionless interaction strength $\tilde{g} = g_{1D}N/(\varepsilon_0 a_0)$, where N denotes the number of atoms in the trap and g_{1D} is the effective 1D interaction strength [25]. To be specific, we use the double-well potential

$$V_{\rm dw} = h\left(\xi^2 - \xi_0^2\right)^2, \qquad (7)$$

where h parametrizes the barrier height between the two wells centered at $\pm\xi_0$. It is straightforward to solve Eq. (6) with the potential (7) and find the lowest symmetric and antisymmetric eigenstates as well as their respective energies E_S and E_A. Figure 2 shows typical results obtained for low and high interactions strengths, respectively. As is apparent from the figure, the higher repulsive interaction strength is associated with more strongly delocalized double-well

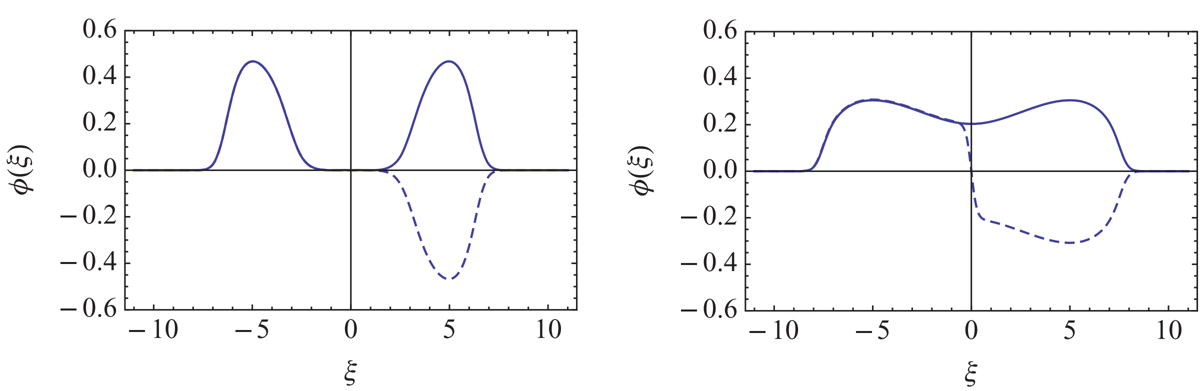

FIG. 2. (Color online) Lowest symmetric (solid curve) and antisymmetric (dashed curve) condensate wave functions obtained for a double-well potential [see Eq. (7)] with $\xi_0 = 5$, $h = 0.05$, and $\tilde{g} = 30$ (left panel) or 300 (right panel). Notice the greater delocalization of atoms between the two wells when the interaction strength is high. This arises because repulsive interactions result in an effective lowering of the tunnel barrier.

wave functions, indicating an effectively stronger tunnel coupling. This can be explained simply by noting that the nonlinear interaction energy for the two condensate fractions in each well shifts up their respective energies, thus effectively lowers the barrier and brings the condensates closer together. As a result, the effective tunnel coupling increases. Most importantly, the energy difference between the lowest symmetric and antisymmetric eigenstates remains positive for any strength of repulsive interactions, which implies that the sign of the tunnel coupling κ entering Eq. (1) is negative.

Several previous works relate to the sign or value of the coupling constant. Milburn *et al.* [7] present an analytical expression for tunnel coupling and the sign is contrary to our findings. The authors of Refs. [8,17,20] do not discuss the sign of the coupling constant explicitly but imply by usage of examples a negative sign (in our convention), which is consistent with our findings. The same sign is implied by calculations of Ananikian and Bergeman [14], which are similar to those in Sec. III, although the sign is not explicitly discussed in Ref. [14].

IV. CONCLUSIONS

The authors of Ref. [21] predict a dynamical instability of a repulsively interacting BEC in a double-ring trap against angular momentum fluctuations. We have carefully revisited the analysis of Ref. [21] and have recalculated the elementary excitation spectrum. This leads us to a different conclusion that makes physical sense. The ground state of a nonrotating condensate in the double-ring configuration is stable against spontaneous angular momentum oscillations. However, the antisymmetric state with its circular node between the two annular quantum wells can be viewed as the analog of a stationary 2D dark soliton, which is known to have a dynamical instability toward the formation of local vorticity (“snake” instability) [22].

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[1] R. Gati and M. K. Oberthaler, J. Phys. B **40**, R61 (2007).
[2] M. Albiez, R. Gati, J. Fölling, S. Hunsmann, M. Cristiani, and M. K. Oberthaler, Phys. Rev. Lett. **95**, 010402 (2005).
[3] Y. Shin, G.-B. Jo, M. Saba, T. A. Pasquini, W. Ketterle, and D. E. Pritchard, Phys. Rev. Lett. **95**, 170402 (2005).
[4] S. Levy, E. Lahoud, I. Shomroni, and J. Steinhauer, Nature (London) **449**, 579 (2007).
[5] B. Eiermann, T. Anker, M. Albiez, M. Taglieber, P. Treutlein, K.-P. Marzlin, and M. K. Oberthaler, Phys. Rev. Lett. **92**, 230401 (2004).
[6] M. Greiner, O. Mandel, T. W. Hänsch, and I. Bloch, Nature (London) **419**, 51 (2002).
[7] G. J. Milburn, J. Corney, E. M. Wright, and D. F. Walls, Phys. Rev. A **55**, 4318 (1997); D. F. Walls and G. J. Milburn, *Quantum Optics*, 2nd ed. (Springer, Heidelberg, 2008).
[8] A. Smerzi, S. Fantoni, S. Giovanazzi, and S. R. Shenoy, Phys. Rev. Lett. **79**, 4950 (1997); S. Raghavan, A. Smerzi, S. Fantoni, and S. R. Shenoy, Phys. Rev. A **59**, 620 (1999).
[9] I. Zapata, F. Sols, and A. J. Leggett, Phys. Rev. A **57**, R28 (1998).
[10] J. Ruostekoski and D. F. Walls, Phys. Rev. A **58**, R50 (1998).
[11] R. W. Spekkens and J. E. Sipe, Phys. Rev. A **59**, 3868 (1999).
[12] D. J. Masiello and W. P. Reinhardt, Phys. Rev. A **76**, 043612 (2007).
[13] S. Giovanazzi, J. Esteve, and M. K. Oberthaler, New J. Phys. **10**, 045009 (2008).
[14] D. Ananikian and T. Bergeman, Phys. Rev. A **73**, 013604 (2006).
[15] J. Javanainen and M. Yu. Ivanov, Phys. Rev. A **60**, 2351 (1999).
[16] L. Pitaevskii and S. Stringari, *Bose-Einstein Condensation* (Clarendon, Oxford, 2003).
[17] M. P. A. Fisher, P. B. Weichman, G. Grinstein, and D. S. Fisher, Phys. Rev. B **40**, 546 (1989).
[18] I. Bouchoule, Eur. Phys. J. D **35**, 147 (2005).
[19] V. M. Kaurov and A. B. Kuklov, Phys. Rev. A **73**, 013627 (2006).
[20] M. Jääskelainen and S. Stenholm, Phys. Rev. A **68**, 033607 (2003).
[21] I. Lesanovsky and W. von Klitzing, Phys. Rev. Lett. **98**, 050401 (2007).
[22] J. Brand and W. P. Reinhardt, Phys. Rev. A **65**, 043612 (2002).
[23] J. Brand, T. J. Haigh, and U. Zülicke, Phys. Rev. A **80**, 011602(R) (2009).
[24] A. Messiah, *Quantum Mechanics*, Vol. I (North-Holland, Amsterdam, 1965), Chap. III, Sec. 12, pp. 109–110.
[25] M. Olshanii, Phys. Rev. Lett. **81**, 938 (1998).