

Decay of superfluid currents in the interacting one-dimensional Bose gas

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We examine the superfluid properties of a one-dimensional (1D) Bose gas in a ring trap based on the model of Lieb and Liniger. While the 1D Bose gas has nonclassical rotational inertia and exhibits quantization of velocities, the metastability of currents depends sensitively on the strength of interactions in the gas: the stronger the interactions, the faster the current decays. It is shown that the Landau critical velocity is zero in the thermodynamic limit due to the first supercurrent state, which has zero energy and finite probability of excitation. We calculate the energy dissipation rate of ring currents in the presence of weak defects, which should be observable on experimental time scales.

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I. INTRODUCTION

Superfluidity is one of the most dramatic manifestations of quantum mechanics on the macroscopic scale and is associated to a host of different phenomena such as nonclassical rotational inertia, quantization of vortices, dragless motion of impurities, and metastability of ring currents as seen in, e.g., liquid He II. Since each of these phenomena may be taken as “defining” a transition to superfluidity, it is important to ask under what circumstances they occur together. As was pointed out by Leggett [1] the metastability of ring currents and nonclassical rotational inertia are two fundamental superfluid phenomena of yet very different nature. While the latter is an equilibrium property, the former is a dynamic one. Although both types of phenomena are often explained by Bose-Einstein condensation of bosons or Cooper pairs of fermions [1], the latter is not seen as an exclusive requirement [2,3]. Here we consider the superfluid properties of an interacting one-dimensional (1D) Bose gas at zero temperature, a system which is not Bose condensed [4,5] but may possess quasi-long-range order [6]. It is a long-standing question whether the 1D Bose gas can support persistent currents with macroscopic lifetimes [5].

This system has been realized with ultracold bosonic atoms in tightly confining linear traps [7,8] (ring traps are also under development [9]), in which the boson interactions are effectively described [10,11] by the contact potential $V(x) = g_B \delta(x)$ of the Lieb-Liniger (LL) model [12]. The interaction strength is quantified by the dimensionless parameter $\gamma = mg_B / (\hbar^2 n)$, where n is the linear density and m is the mass. For $\gamma \rightarrow \infty$, the model is known as the Tonks-Girardeau (TG) gas and can be mapped to an ideal Fermi gas. For $\gamma \ll 1$, the Bogoliubov model of weakly interacting bosons is recovered.

Experimental investigation of the superfluid properties of the 1D Bose gas by observing the motion of impurities is at an early stage [8] and theoretical predictions are not yet comprehensive. Sonin [2] found that ring currents can be metastable except for infinitely strong interactions. Kagan *et al.* [13] also concluded that persistent currents could be observ-

able on experimental time scales and Büchler *et al.* [14] found the 1D Bose gas able to sustain supercurrents even in the presence of a strong defect. Astrakharchik and Pitaevskii [15] considered the drag force on a moving heavy impurity within Luttinger liquid theory and predicted a power-law dependence on the velocity for small velocities. These results contain an unknown prefactor preventing the calculation of the actual value of the drag force and are in any case not applicable at larger velocities. The motion of an impurity of finite mass was considered in the TG gas [16] but for finite values of γ this problem is still unresolved.

In this paper we calculate the rate of energy dissipation of ring currents in the presence of a small integrability-breaking defect of strength g_i based on recent advances in the understanding [17–19] of the dynamics of the LL model. The results of our calculations are summarized in Fig. 1. While for small velocities our calculations support the power-law predictions of Ref. [15], the drag force $F_v = 2g_i^2 nm \hbar^{-2} f_v$ assumes the velocity-independent value of $2g_i^2 nm / \hbar^2$ for velocities large compared to the speed of sound c . Although our results

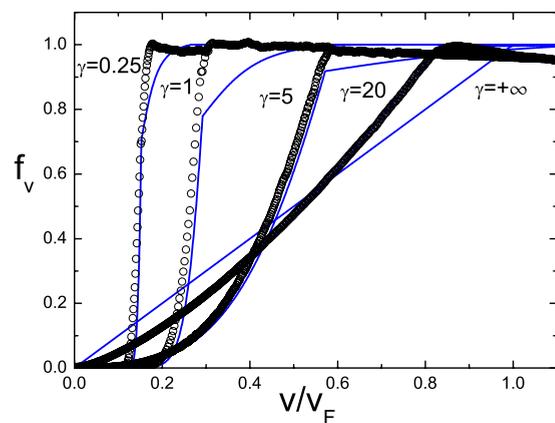


FIG. 1. (Color online) The dimensionless drag force versus the velocity (relative to $v_F = \hbar \pi n / m$) of the impurity at various values of the coupling parameter. The solid (blue) lines represent the force obtained with Eqs. (1) and (4); open circles are the numerical data obtained using ABACUS [18].

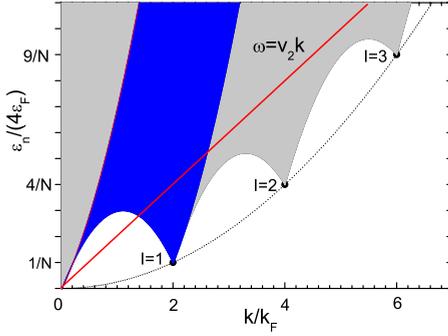


FIG. 2. (Color online) Schematic of the excitation spectrum of the 1D Bose gas in a perfectly isotropic ring. The supercurrent states I lie on the parabola $\hbar^2 k^2 / (2Nm)$ (dotted line). Excitations occur in the shaded area; the discrete structure of the spectrum is not shown for simplicity. The blue (dark) area represents particle-hole excitations [21]. Motion of the impurity with respect to the gas causes transitions from the ground state to the states lying on the straight (red) line.

suggest that the 1D Bose gas can support metastable currents only in the weakly interacting regime where $\gamma \ll 1$, the superfluid fraction is 1 regardless of γ [20] according to the nonclassical rotational inertia for a finite ring.

II. LANDAU CRITERION OF SUPERFLUIDITY

In the LL model the total momentum is a good quantum number and periodic boundary conditions quantize it in units of $2\pi\hbar/L$, where L is the ring circumference. The low-lying spectrum of $N=nL$ bosons as shown in Fig. 2 has local minima [22] at the supercurrent states I ($I=0, 1, 2, \dots$) with momenta $p_I = 2\pi\hbar I$ and excitation energies $\varepsilon_I = p_I^2 / (2Nm)$. These correspond to Galilean transformations of the ground state with velocities $v_I = p_I / (Nm)$. The minima do not depend on interactions and tend to zero in the limit of large system size.

Suppose that the gas is initially rotating with the linear velocity $-v_I$ and then is braked with an “obstacle,” created, e.g., by a laser beam [23]. In the frame where the gas is at rest, the obstacle moves with velocity v_I . In a superfluid we expect no energy dissipation and thus zero drag force (the current is persistent). Energy conservation dictates that the transitions from the ground state caused by the moving obstacle with velocity v lie on the line $\varepsilon = vp$. According to Landau, if the excitation spectrum lies above this line, the motion cannot excite the system, which is then regarded superfluid. The Landau critical velocity (when the line touches the spectrum) equals $v_c = \varepsilon_I / p_I = v_I / 2$. This implies that any supercurrent state with $I \geq 1$ is unstable since $v_I > v_c$. However, in 3D similar supercurrent states exist, which apparently leads to the absence of current metastability. The paradox can be resolved by considering not only the spectrum but also *probabilities* of excitations. Below we argue that in the 3D case, the probability to excite supercurrents is vanishingly small, while in the 1D case it depends on the strength of bosonic interactions.

III. HESS-FAIRBANK EFFECT

When the walls of a toroidal container are set in rotation adiabatically with a small velocity, a superfluid stays at rest

while a normal fluid follows the container. This effect leads to a nonclassical rotational inertia of superfluid systems, which can be used to determine the superfluid fraction [3]. One can show [20] that in the LL model, the gas has zero rotational inertia (zero normal fraction) at $T=0$ for any $\gamma > 0$. This is an *equilibrium* property completely determined by the low-lying energy spectrum [1].

IV. DECAY OF SUPERCURRENTS

A. Dynamic response

By contrast to the Hess-Fairbank effect, metastability of currents is not an equilibrium effect and transition probabilities have to be considered. The dissipation rate as energy loss per unit time \dot{E} of an obstacle (or heavy impurity) moving with velocity v relative to the gas can be related to the drag force F_v acting on the impurity by $\dot{E} = -F_v v$. For weak impurities with interaction potential $V_i(x)$ the drag force is related to the dynamic structure factor (DSF) in linear-response theory [15,24]:

$$F_v(v) = \int_0^{+\infty} dk |\tilde{V}_i(k)|^2 S(k, kv) / L, \quad (1)$$

where $\tilde{V}_i(k)$ is the Fourier transform of the impurity potential. The DSF $S(k, \omega)$ describes the transition probability between the ground state $|0\rangle$ and excited states $|m\rangle$ with energy transfer $\hbar\omega$ and momentum transfer $\hbar k$ caused by a density perturbation and can be written as

$$S(k, \omega) = \sum_m |\langle 0 | \delta \hat{p}_k | m \rangle|^2 \delta(\hbar\omega - E_m + E_0), \quad (2)$$

where $\delta \hat{p}_k = \sum_j e^{-ikx_j} - N\Delta(k)$ is the Fourier component of the density operator, $\Delta(k) = 1$ at $k=0$ and $\Delta(k) = 0$ otherwise. Several results for the DSF in the LL model have recently become available [17–19]. It can be measured in cold gases by Bragg scattering [25,26].

Numerical values of DSF calculated with the ABACUS algorithm [18] are shown in Fig. 3. The probability to create multiparticle excitations lying outside of the region $\omega_-(k) \leq \omega \leq \omega_+(k)$ are identically zero (below ω_-) or very small (above ω_+). Transitions from the ground state caused by a moving obstacle with velocity v occur along the straight (red) line. The drag force (1) is thus a generalization of the Landau criterion for superfluidity. Indeed, if the excitation spectrum of a generic system lies above the line $\omega = vk$ then it is superfluid; in this case the drag force (1) equals zero. The drag force thus proves to be fundamental and can be considered as a quantitative measure of superfluidity.

B. Shallow optical lattices

Equation (1) can be verified experimentally for different types of obstacles: for $V_i(x) = g_i \delta(x)$ all the points at the line contribute to the drag force, while for $V_i(x) = g_L \cos(2\pi x/a)$ only one point $(k_G, k_G v)$ in the k - ω plane does, where $k_G \equiv 2\pi/a$ is the reciprocal lattice vector (see Fig. 3). Indeed, substituting the Fourier transform into Eq. (1) yields

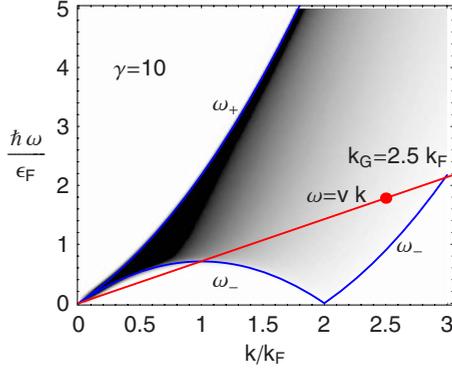


FIG. 3. (Color online) Dynamic structure factor of the 1D Bose gas from [18] for $N=100$. Dimensionless values of $S(k, \omega)\epsilon_F/N$ are shown in shades of gray between 0 (white) and 0.7 (black). The full (blue) lines represent the limiting dispersion relations ω_{\pm} and the straight (red) line is the line of integration in Eq. (1). Only one point at $k=k_G$, shown in full (red) circle, contributes to the integral when the perturber is a shallow cosine potential with a reciprocal vector k_G .

$$F_v = \pi g_L^2 k_G S(k_G, k_G v) / 2. \quad (3)$$

The filling factor of the lattice is given by $2\pi n/k_G$. Equation (3) can be exploited even in the case of a cigar-shaped quasi-1D gas of bosons at large number of particles, because the boundary conditions do not play a role in the thermodynamic limit. It gives us the momentum transfer per unit time from a moving shallow lattice, which can be measured experimentally [27]. At $k_G=2\pi n$, corresponding to the Mott insulator state in a deep lattice, and at $\gamma \gg 1$, the drag force takes nonzero values for arbitrary $v \leq \omega_+(k_G)/k_G$. However, at small γ , its nonzero values practically localize in vicinity of $v = \omega_+(k_G)/k_G$. As there is no sharp transition from superfluid to isolated phase in 1D [28], we can put the threshold equal to, say, 0.1 of the characteristic value $\pi g_L^2 k_G N / (8\epsilon_F)$ of drag force (3). Then we get a phase diagram in the v - γ plane [29] similar to that of Polkovnikov *et al.* [28]. Note that in the latter paper, the superfluidity was examined in terms of quantum phase slips [30]. So, the both quasiparticle and quantum phase slip description lead to the same results.

C. Approximate expression for drag force

In order to study metastability of the l th supercurrent state, we need to calculate the drag force on an obstacle moving with the velocity v_l relative to the gas. For large system size the supercurrent-state velocities are dense and in the thermodynamic limit ($N \rightarrow \infty, n = \text{const}$) we may consider arbitrary velocities. We consider the drag force and decay of currents in various regimes.

We calculate the drag force from Eq. (1) by using the interpolating expression

$$S(k, \omega) = C(\omega^\alpha - \omega_-^\alpha)^{\mu_-} / (\omega_+^\alpha - \omega^\alpha)^{\mu_+} \quad (4)$$

for $\omega_-(k) \leq \omega \leq \omega_+(k)$, and $S(k, \omega) = 0$ otherwise [31]. Here, $K \equiv \hbar \pi n / (mc)$ is the Haldane parameter [15], $\mu_+(k)$ and $\mu_-(k)$ are the exact exponents [19] at the borders of the spectrum $\omega_+(k)$ and $\omega_-(k)$, and $\alpha \equiv 1 + 1/\sqrt{K}$. The values of

$\omega_{\pm}(k)$ and $\mu_{\pm}(k)$ can be calculated from the coupling constant γ numerically by the methods outlined in Refs. [19,21]. The normalization constant C depends on momentum but not on frequency and is determined from the f -sum rule $\int_{-\infty}^{+\infty} d\omega \omega S(q, \omega) = Nq^2 / (2m)$. Expression (4) is applicable for all ranges of the parameters k , ω , and γ with increasing accuracy at large γ . A more detailed discussion can be found in Ref. [31].

D. Numerical results

We further restrict ourselves to a δ -function impurity interaction with $\tilde{V}_i(k) = g_i$. Results of integrating Eq. (1) are shown in Fig. 1. For large velocities the drag force reaches the velocity-independent value of $2g_i^2 nm / \hbar^2$. A characteristic velocity scale is the speed of sound c , which determines the transition from a power-law increase to the velocity-independent regime. The speed of sound c of the LL model is proportional to γ for small γ but saturates to the value v_F for large γ [12]. The numerical DSF as per Ref. [18] was obtained for $N=150$ particles ($\gamma=20, 5$), $N=200$ ($\gamma=1$), and $N=300$ ($\gamma=0.25$). The f -sum rule saturations at $k=2k_F$ were 99.64% ($\gamma=20$), 97.81% ($\gamma=5$), 99.06% ($\gamma=1$), and 99.08% ($\gamma=0.25$), with yet better results at smaller momentum. The fit with the analytical ansatz is good for all values of v for large γ . The decreasing curves at large velocity $v \gg c$ are due to imperfect sum-rule saturation at high momenta. For small γ , the onset of the drag force is quicker from the numerical DSF than from the analytical ansatz. This occurs first because the smoothing of the numerical data required to compute the drag force overestimates it when its curvature is positive (this smoothing also leads to small artifacts in the data around $v=c$), and second because the obtained numerical DSF is larger than the analytical ansatz for $\omega \ll \omega_+$, and also just above the Bogoliubov dispersion (where the analytical ansatz is zero by definition), where excitations with higher numbers of particle-hole pairs contribute.

E. Drag force at small velocities

For the important question whether persistent currents may exist at all, the small velocity regime is most relevant, which is dominated by transitions near the first supercurrent state (umklapp point at $k=2k_F$). The drag force in this regime has a power-law dependence on the velocity $F_v \sim v^{2K-1}$ for $v \ll c$, as first found by Astrakharchik and Pitaevskii. From Eqs. (1) and (4) we can obtain

$$\begin{aligned} f_v &\equiv \frac{F_v \pi \epsilon_F}{g_i^2 k_F^3} \\ &\simeq 2K \left(\frac{v}{v_F} \right)^{2K-1} \left(\frac{4\epsilon_F}{\hbar \omega_+(2k_F)} \right)^{2K} \\ &\quad \times \frac{\Gamma\left(1 + \frac{2K}{\alpha} - \mu_+(2k_F)\right) \Gamma(1 + \mu_-(2k_F)) \Gamma\left(1 + \frac{1}{\alpha}\right)}{\Gamma\left(\frac{2K}{\alpha}\right) \Gamma(1 - \mu_+(2k_F)) \Gamma\left(1 + \mu_-(2k_F) + \frac{1}{\alpha}\right)}, \end{aligned} \quad (5)$$

where $\Gamma(x)$ is Euler's gamma function and $\mu_-(2k_F) = 2\sqrt{K}(\sqrt{K}-1)$ [19]. This formula is valid for *arbitrary* coupling constant and works even in the Bogoliubov regime at $\gamma \ll 1$. In practice, Eq. (5) works well up to $v \leq 0.1c$.

F. Why excitations near the umklapp point do not play a role in three dimensions

The behavior of the DSF near the umklapp point means that the drag force takes nonzero values even for *arbitrarily small* interactions. This fact is related to the absence of Bose-Einstein condensation in the 1D Bose gas. For large interactions, umklapp excitations become readily available and provide an avenue for the rapid decay of supercurrents. Landau reasoned that this is very implausible in 3D, since it involves the macroscopic motion of the system, and hence a *macroscopic number* of quasiparticle excitations. Strong correlations in 1D, however, make umklapp excitations easily accessible, since they involve only a *single* fermionlike quasiparticle [12].

G. Currents in a ring

In the presence of an obstacle a ring current can decay into supercurrent states with smaller momentum. Starting in one of the local minima of the spectrum in Fig. 2, the kinetic energy of the center of mass will be transformed into elementary excitations above a lower supercurrent state conserving the total energy. The elementary excitations are quasiparticle-quasihole excitations in the Bethe-ansatz wave function [21] and may have mixed phonon and soliton character. Assuming that these have little effect on successive transitions, we estimate the decay of the center-of-mass velocity v by the classical equation $Nm\dot{v} = -F_v(v)$, where F_v is given by Eqs. (1) and (4). This was integrated numerically and the result is shown in Fig. 4. At the initial supersonic velocity, where the drag force is saturated (see Fig. 1) the supercurrent experiences constant deceleration. For $v \lesssim c$ the drag force decreases and the deceleration slows down. For the TG gas we find an analytical solution for exponential

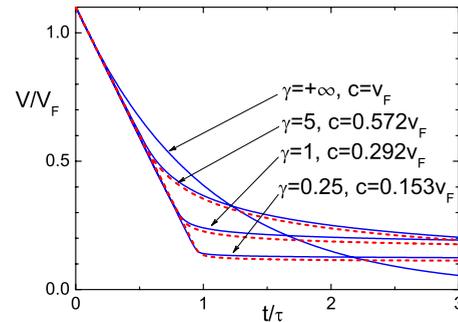


FIG. 4. (Color online) Decay of the ring current velocity of 1D bosons from the initial velocity of $1.1v_F$ at $t=0$. The solid (blue) and dashed (red) lines represent the results obtained with the approximate formula and ABACUS, respectively. The time scale is $\tau = N\pi\hbar^3/(2mg_1^2)$.

decay $v(t) = v_0 \exp(-t/\tau)$ for $v_0 \leq v_F$. In the weakly interacting regime, the decay becomes slow compared to experimental time scales.

V. CONCLUSION

Concluding, although the 1D Bose gas with finite repulsive interparticle interaction shows superfluid phenomena of the *equilibrium type*, we show that in general its ability to support *dynamic* superfluid phenomena such as persistent ring currents is limited to a regime of very weak interactions; for a periodic potential, braking the gas, the persistent currents can be observed even in the TG regime at specific values of the velocity and density.

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- [1] A. J. Leggett, Rev. Mod. Phys. **71**, S318 (1999).
 [2] É. B. Sonin, Sov. Phys. JETP **32**, 773 (1970); Sov. Phys. Usp. **25**, 409 (1982).
 [3] A. J. Leggett, Phys. Fenn. **8**, 125 (1973).
 [4] N. N. Bogoliubov, *Quasiexpectation Values in Problems of Statistical Mechanics* (Gordon and Breach, New York, 1961).
 [5] P. C. Hohenberg, Phys. Rev. **158**, 383 (1967).
 [6] D. S. Petrov, G. V. Shlyapnikov, and J. T. M. Walraven, Phys. Rev. Lett. **85**, 3745 (2000).
 [7] T. Kinoshita, T. Wenger, and D. S. Weiss, Science **305**, 1125 (2004).
 [8] S. Palzer, C. Zipkes, C. Sias, and M. Köhl, e-print arXiv:0903.4823.
 [9] S. Gupta, K. W. Murch, K. L. Moore, T. P. Purdy, and D. M. Stamper-Kurn, Phys. Rev. Lett. **95**, 143201 (2005); O. Moritz, Y. Colombe, V. Lorent, H. Perrin, and B. M. Garraway, Phys. Rev. A **74**, 023617 (2006); C. Ryu, M. F. Andersen, P. Cladé, V. Natarajan, K. Helmerson, and W. D. Phillips, Phys. Rev. Lett. **99**, 260401 (2007); S. E. Olson, M. L. Terraciano, M. Bashkansky, and F. K. Fatemi, Phys. Rev. A **76**, 061404(R) (2007).
 [10] M. Olshanii, Phys. Rev. Lett. **81**, 938 (1998).
 [11] A. Yu. Cherny and J. Brand, Phys. Rev. A **70**, 043622 (2004).
 [12] E. H. Lieb and W. Liniger, Phys. Rev. **130**, 1605 (1963).
 [13] Y. Kagan, N. V. Prokof'ev, and B. V. Svistunov, Phys. Rev. A **61**, 045601 (2000).
 [14] H. P. Büchler, V. B. Geshkenbein, and G. Blatter, Phys. Rev. Lett. **87**, 100403 (2001).
 [15] G. E. Astrakharchik and L. P. Pitaevskii, Phys. Rev. A **70**, 013608 (2004).

- [16] M. D. Girardeau and A. Minguzzi, *Phys. Rev. A* **79**, 033610 (2009).
- [17] J. Brand and A. Yu. Cherny, *Phys. Rev. A* **72**, 033619 (2005); A. Yu. Cherny and J. Brand, *ibid.* **73**, 023612 (2006).
- [18] J.-S. Caux and P. Calabrese, *Phys. Rev. A* **74**, 031605(R) (2006); J.-S. Caux, *J. Math. Phys.* **50**, 095214 (2009).
- [19] A. Imambekov and L. I. Glazman, *Phys. Rev. Lett.* **100**, 206805 (2008); *Science* **323**, 228 (2009).
- [20] M. Ueda and A. J. Leggett, *Phys. Rev. Lett.* **83**, 1489 (1999); M. A. Cazalilla, *J. Phys. B* **37**, S1 (2004).
- [21] E. H. Lieb, *Phys. Rev.* **130**, 1616 (1963).
- [22] F. D. M. Haldane, *Phys. Rev. Lett.* **47**, 1840 (1981).
- [23] C. Raman, M. Köhl, R. Onofrio, D. S. Durfee, C. E. Kuklewicz, Z. Hadzibabic, and W. Ketterle, *Phys. Rev. Lett.* **83**, 2502 (1999).
- [24] E. Timmermans and R. Côté, *Phys. Rev. Lett.* **80**, 3419 (1998).
- [25] J. Stenger, S. Inouye, A. P. Chikkatur, D. M. Stamper-Kurn, D. E. Pritchard, and W. Ketterle, *Phys. Rev. Lett.* **82**, 4569 (1999).
- [26] R. Ozeri, N. Katz, J. Steinhauer, and N. Davidson, *Rev. Mod. Phys.* **77**, 187 (2005).
- [27] L. Fallani, L. De Sarlo, J. E. Lye, M. Modugno, R. Saers, C. Fort, and M. Inguscio, *Phys. Rev. Lett.* **93**, 140406 (2004); C. D. Fertig, K. M. O'Hara, J. H. Huckans, S. L. Rolston, W. D. Phillips, and J. V. Porto, *ibid.* **94**, 120403 (2005); J. Mun, P. Medley, G. K. Campbell, L. G. Marcassa, D. E. Pritchard, and W. Ketterle, *ibid.* **99**, 150604 (2007).
- [28] A. Polkovnikov, E. Altman, E. Demler, B. Halperin, and M. D. Lukin, *Phys. Rev. A* **71**, 063613 (2005); J. Ruostekoski and L. Isella, *Phys. Rev. Lett.* **95**, 110403 (2005); A. R. Kolovsky, *New J. Phys.* **8**, 197 (2006).
- [29] A. Yu. Cherny, J.-S. Caux, and J. Brand (unpublished).
- [30] K. Arutyunov, D. Golubev, and A. Zaikin, *Phys. Rep.* **464**, 1 (2008).
- [31] A. Yu. Cherny and J. Brand, *J. Phys.: Conf. Ser.* **129**, 012051 (2008); *Phys. Rev. A* **79**, 043607 (2009).