Dynamic and static density-density correlations in the one-dimensional Bose gas: Exact results and approximations

Alexander Yu. Cherny[1] and Joachim Brand[2]

[1]*Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, 141980 Dubna, Moscow region, Russia*

[2]*Centre for Theoretical Chemistry and Physics and Institute of Natural Sciences, Massey University, Private Bag 102 904, North Shore, Auckland 0745, New Zealand*

We discuss approximate formulas for the dynamic structure factor of the one-dimensional Bose gas in the Lieb-Liniger model that appear to be applicable over a wide range of the relevant parameters such as the interaction strength, frequency, and wave number. The suggested approximations are consistent with the exact results known in limiting cases. In particular, we encompass exact edge exponents as well as the Luttinger-liquid and perturbation theoretic results. We further discuss derived approximations for the static structure factor and the pair distribution function $g(x)$. The approximate expressions show excellent agreement with numerical results based on the algebraic Bethe ansatz.

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I. INTRODUCTION

Correlations in ultracold atomic gases arise from the interplay of quantum statistics, interactions, and thermal and quantum fluctuations. Recently, a lot of progress has been made experimentally to probe and characterize these correlations [1–4]. The one-dimensional (1D) Bose gas is a particularly interesting system as quantum correlations generally play a larger role compared to three-dimensional Bose-Einstein condensates, and regimes with very different correlation properties can be probed experimentally [5,6]. In these experiments, elongated "spaghetti" traps are created by optical lattices, which confine the atomic motion in the transverse dimensions to zero-point quantum oscillations [7]. Thus, the systems become effectively one dimensional.

Theoretically, interactions of the rarefied atoms in one-dimensional waveguides are well described by effective δ-function interactions [8,9]. The resulting model of a one-dimensional Bose gas is an archetype of an integrable but nontrivial many-body system that has been receiving long-standing interest from physicists and mathematicians alike. The model was first solved with Bethe ansatz by Lieb and Liniger [10,11], who calculated the ground-state and excitation energies. Depending on the value of the dimensionless coupling strength, the Lieb-Liniger model describes various regimes with the corresponding correlations. Being exactly solvable, the model, however, does not admit complete analytic solution for the correlation functions. Up to now, this is a complicated and challenging problem in 1D physics [12–14].

The aim of this paper is to provide practical approximate formulas for various density correlation functions of the one-dimensional Bose gas. The main results are contained in expressions (13), (19), (22), and (27) for the dynamic and static structure factors and the dynamic polarizability, respectively. These formulas interpolate exact results, while prefactors and parameters are fixed completely by requiring consistency with the known large-interaction expansion and the f-sum rule. Favorable comparison with numerical results validates the interpolation formulas for practical use.

Dynamical density-density correlations can be measured in cold atoms by the two-photon Bragg scattering [1,15]. Theoretically, they are described by the dynamic structure factor (DSF) [16],

$$S(k,\omega) = L \int \frac{dt\,dx}{2\pi\hbar} e^{i(\omega t - kx)} \langle 0|\delta\hat{\rho}(x,t)\delta\hat{\rho}(0,0)|0\rangle, \quad (1)$$

where $\delta\hat{\rho}(x,t) \equiv \hat{\rho}(x,t) - n$ is the operator of density fluctuations and $n = N/L$ is the equilibrium density of particles. We consider zero temperature, where $\langle 0|\ldots|0\rangle$ denotes the ground-state expectation value. The DSF is proportional to the probability of exciting a collective mode from the ground state with the transfer of momentum k and energy $\hbar\omega$, as can be seen from the energy representation of Eq. (1),

$$S(k,\omega) = \sum_m |\langle 0|\delta\hat{\rho}_k|n\rangle|^2 \delta(\hbar\omega - E_m + E_0), \quad (2)$$

where $\delta\hat{\rho}_k = \sum_j e^{-ikx_j} - N\Delta(k)$ is the Fourier component of $\delta\hat{\rho}(x)$, $\Delta(k) = 1$ at $k=0$ and $\Delta(k)=0$ otherwise. Once the DSF is known, the static structure factor $S(k)$ and the pair distribution function $g(x)$ can be calculated by integration as is discussed in Sec. III B.

Previously known results for the DSF of the one-dimensional Bose gas come from Luttinger-liquid theory, which predicts a power-law behavior of the DSF at low energies in the vicinity of the momenta $k=0, 2\pi n, 4\pi n\ldots$ and yields universal values for the exponents [17–19]. In the regime of strong interactions, we have previously derived perturbatively valid expressions covering arbitrary energies and momenta at zero [20] and finite temperature [21]. For finite systems, it is possible to compute the correlation functions numerically using the results of the algebraic Bethe ansatz calculations [22,23]. Finally, the exact power-law behavior along the limiting dispersion curve of the collective modes has recently been calculated in Refs. [24,25]. These exponents differ from those predicted by Luttinger-liquid theory, raising the question whether the different results are compatible with each other. We address this question in Sec. III A of this paper, where we show that the results can be reconciled

by taking appropriate limits. The apparent difference between the edge exponents valid along the dispersion curves and the Luttinger-liquid result in the limit of vanishing energy can be traced back to the fact that the dispersion relations are curved and not straight, as is presumed by Luttinger-liquid theory.

The exact values of the exponents found in Refs. [24,25] are of importance; however, they are not sufficient for practical estimations of the DSF as long as the prefactors are not known. In this paper we construct an approximate formula for the DSF [26] based on the exponents of Refs. [24,25]. Within the proposed scheme, the prefactor can be found using the well-known f-sum rule (see, e.g., [16]). The result turns out to be consistent with numerical results by Caux and Calabrese [22]. Besides, it is compatible with the results of Luttinger-liquid theory [17–19] and perturbation theory [20]. The approximate formula, in effect, takes into account single quasiparticle-quasihole excitations but neglects multiparticle excitations. We also present an approximate expression for the static structure factor and for the density-density correlation function, which is derived from the approximation for the DSF.

II. EXACT RESULTS FOR DYNAMIC STRUCTURE FACTOR IN LIEB-LINIGER MODEL

We model cold bosonic atoms in a waveguidelike microtrap by a simple 1D gas of N bosons with point interactions of strength $g_B > 0$,

$$H = \sum_{i=1}^{N} -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_i^2} + g_B \sum_{1 \le i < j \le N} \delta(x_i - x_j), \quad (3)$$

and impose periodic boundary conditions on the wave functions. The strength of interactions can be measured in terms of dimensionless parameter $\gamma = mg_B/(\hbar^2 n)$. In the limit of large γ, the model is known as the Tonks-Girardeau (TG) gas. In this limit, it can be mapped onto an ideal *Fermi* gas since infinite contact repulsions emulate the Pauli principle. In the opposite limit of small γ, we recover the Bogoliubov model of weakly interacting bosons.

A. DSF expansion in $1/\gamma$

For finite γ, the model can also be mapped onto a Fermi gas [27] with local interactions, inversely proportional to g_B [20,21,28,29]. Using the explicit form of the interactions, one can develop the time-dependent Hartree-Fock scheme [20,21] in the strong-coupling regime with small parameter $1/\gamma$. The scheme yields the correct expansion of the DSF up to the first order [20,21]

$$S(k,\omega)\frac{\varepsilon_F}{N} = \frac{k_F}{4k}\left(1 + \frac{8}{\gamma}\right) + \frac{1}{2\gamma}\ln\frac{\omega^2 - \omega_-^2}{\omega_+^2 - \omega^2} + O\left(\frac{1}{\gamma^2}\right) \quad (4)$$

for $\omega_-(k) \le \omega \le \omega_+(k)$, and zero elsewhere [30]. The symbol $O(x)$ denotes terms of order x or even smaller. Here $\omega_\pm(k)$ are the limiting dispersions [31] that bound quasiparticle-quasihole excitations (see Fig. 1); in the strong-coupling regime they take the form
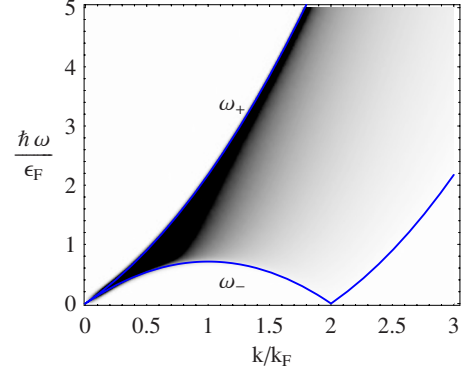

FIG. 1. (Color online) Numerical values of the DSF [Eq. (2)] for the coupling parameter $\gamma = 10$ [22]. The dimensionless value of the rescaled DSF $S(k,\omega)\varepsilon_F/N$ is shown in shades of gray between zero (white) and 1.0 (black). The upper and lower solid (blue) lines represent the dispersions $\omega_+(k)$ and $\omega_-(k)$, respectively, limiting the single "particle-hole" excitations in the Lieb-Liniger model at $T = 0$. The dispersions are obtained numerically by solving the system of integral equations [11]. The gray scale plot of the DSF demonstrates that the main contribution to the DSF comes from the single particle-hole excitations lying inside the region $\omega_-(k) \le \omega \le \omega_+(k)$ (see also Fig. 3).

$$\omega_\pm(k) = \hbar|2k_F k \pm k^2|(1 - 4/\gamma)/(2m) + O(1/\gamma^2). \quad (5)$$

By definition, $k_F \equiv \pi n$ and $\varepsilon_F \equiv \hbar^2 k_F^2/(2m)$ are the Fermi wave vector and energy of TG gas, respectively.

B. Link to Luttinger-liquid theory

Luttinger-liquid theory describes the behavior of the DSF at low energies for arbitrary strength of interactions [17,19]. In particular, one can show [18,19] that in the vicinity of "umklapp" point ($k = 2\pi n, \omega = 0$) it is given by

$$\frac{S(k,\omega)}{N} = \frac{nc}{\hbar\omega^2}\left(\frac{\hbar\omega}{mc^2}\right)^{2K}A(K)\left(1 - \frac{\omega_-^2(k)}{\omega^2}\right)^{K-1} \quad (6)$$

for $\omega \ge \omega_-(k)$, and zero otherwise. Within the Luttinger-liquid theory, the dispersion is *linear* near the umklapp point: $\omega_-(k) \simeq c|k - 2\pi n|$. By definition,

$$K \equiv \hbar\pi n/(mc), \quad (7)$$

and c is the sound velocity. For the repulsive bosons, the value of parameter K lies between 1 (TG gas) and $+\infty$ (ideal Bose gas). Discussion of various limiting cases in the Lieb-Liniger model can be found in [10,11,32]. In the strong-coupling regime, the linear behavior of the dispersions [Eq. (5)] at small momentum determines the sound velocity, which allows us to calculate the value of the Luttinger parameter

$$K = 1 + 4/\gamma + O(1/\gamma^2). \quad (8)$$

The coefficient $A(K)$ is model dependent; in the Lieb-Liniger model, it is known in two limiting cases: $A(K) = \pi/4$ at $K = 1$ and $A(K) \simeq 8^{1-2K}\exp(-2\gamma_c K)\pi^2/\Gamma^2(K)$ for $K \gg 1$ [19], where $\gamma_c = 0.5772\ldots$ is the Euler constant and $\Gamma(K)$ is the gamma function.
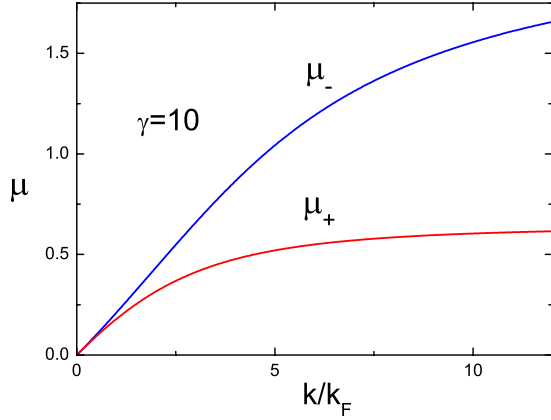
FIG. 2. (Color online) Typical behavior of the exact exponents in Eq. (11). The diagram shows $\mu_\pm$ for $\gamma=10$ obtained numerically using the method of Ref. [25].

By comparing the first-order expansion [Eq. (4)] in the vicinity of the umklapp point with Eq. (6) and using expansion (8), one can easily obtain the model-dependent coefficient at large but *finite* interactions when $K-1 \ll 1$;

$$A(K) = \frac{\pi}{4}[1 - (1 + 4\ln 2)(K-1)] + O((K-1)^2). \quad (9)$$

Note that relation (6) leads to different exponents precisely at the umklapp point and outside of it:

$$S(k,\omega) \sim \begin{cases} \omega^{2(K-1)}, & k = 2\pi n, \\ (\omega - \omega_-)^{K-1}, & k \neq 2\pi n. \end{cases} \quad (10)$$

C. Exact edge exponents from the Lieb-Liniger solutions

As was shown in Refs. [24,25] (see also [33]), within the Lieb-Liniger model the DSF exhibits the following power-law behavior near the borders of the spectrum $\omega_\pm(k)$:

$$S(k,\omega) \sim |\omega - \omega_\pm(k)|^{\mp\mu_\pm(k)}. \quad (11)$$

The positive exponents $\mu_\pm$ [31] are related to the quasiparticle scattering phase and can be calculated in the thermodynamic limit by solving a system of integral equations [25]. In particular, Imambekov and Glazman [25] found the following right limit,

$$\lim_{k\to 2\pi n^-} \mu_-(k) = 2\sqrt{K}(\sqrt{K}-1), \quad (12)$$

which is different from the Luttinger-liquid exponent [Eq. (10)]. However, Imambekov's and Glazman's result [Eq. (12)] is accurate in the immediate vicinity of ω_- provided that the finite curvature of $\omega_-(k)$ is taken into consideration. Thus the difference in the exponents can be attributed [25] to the linear spectrum approximation within the Luttinger-liquid theory. Note, however, that the thin "strip" in ω-k plane, where the exponents are different, vanishes in point $k=2\pi n$; hence, the Luttinger exponent $2(K-1)$ becomes exact there.

A typical behavior of the exponents is shown in Fig. 2. As

described in Ref. [25], the exponents can be easily evaluated by solving an integral equation for the shift function [12].

D. Algebraic Bethe ansatz

Recent progress in the computation of correlation functions within the Lieb-Liniger model and other 1D models has been achieved through the algebraic Bethe ansatz [22]. In this method, matrix elements of the density operator involved in Eq. (2) were calculated with the algebraic Bethe ansatz. They are given by the determinant of a matrix, which can be evaluated numerically for a finite number of particles. So, this method is based on combining integrability and numerics. The results of the numerical calculations of Ref. [22] are shown in Figs. 1 and 3.

III. APPROXIMATE EXPRESSION FOR DYNAMIC STRUCTURE FACTOR

A. Approximate expression for arbitrary values of interaction strength

Here we suggest a phenomenological expression, which is consistent with all the above-mentioned results. It reads as

$$S(k,\omega) = C\frac{(\omega^\alpha - \omega_-^\alpha)^{\mu_-}}{(\omega_+^\alpha - \omega^\alpha)^{\mu_+}} \quad (13)$$

for $\omega_-(k) \leq \omega \leq \omega_+(k)$, and zero otherwise. It follows from energy and momentum conservation that $S(k,\omega)$ is exactly equal to zero below $\omega_-(k)$ for $0 \leq k \leq 2\pi n$. In the other regions of $\omega > \omega_+$ and $\omega < \omega_-$ (for $k > 2\pi n$), possible contributions can arise due to coupling to multiparticle excitations [11]. However, these contributions are known to vanish in the Tonks-Girardeau ($\gamma \to \infty$) and Bogoliubov ($\gamma \to 0$) limits and are found to be very small numerically for finite interactions [22].

The exponents $\mu_\pm$ are non-negative (see Fig. 2). As a consequence, the DSF diverges at the upper branch ω_+. At the lower branch ω_-, the DSF shows a continuous transition to zero for any finite value of γ except for the specific point $\gamma=+\infty$ (or $K=1$) of the Tonks-Girardeau gas, where the DSF remains finite but has a discontinuous transition to zero at both boundaries ω_- and ω_+. Thus, the ω_- branch is suppressed in the DSF for finite γ, and transitions into these excitations will not be seen within linear-response theory.

In Eq. (13) C is a normalization constant, $\mu_+(k)$ and $\mu_-(k)$ are the exponents of Eq. (11), and $\alpha \equiv 1 + 1/\sqrt{K}$. From the definition of K [Eq. (7)], one can see that for repulsive spinless bosons $K \geq 1$, and hence, $1 < \alpha \leq 2$. The normalization constant depends on the momentum but not the frequency and can be determined from the f-sum rule [16]

$$\int_0^{+\infty} d\omega\, \omega S(k,\omega) = N\frac{k^2}{2m}. \quad (14)$$

In Eq. (13) we assume that the value of the exponent $\mu_-(k=2\pi n)$ coincides with its limiting value [Eq. (12)] in vicinity of the umklapp point.

The most general way of obtaining $\omega_\pm(k)$, $\mu_\pm(k)$, and K is to solve numerically the corresponding integral equations
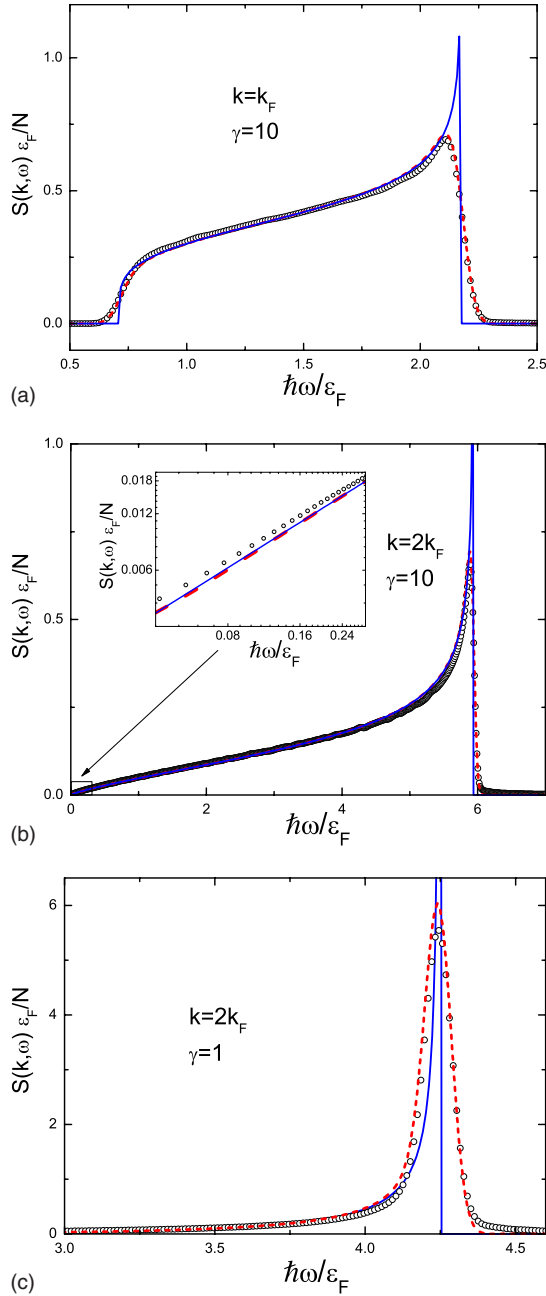
FIG. 3. (Color online) DSF in the thermodynamic limit. The approximation [Eq. (13)] (line) is compared to numerical data from Caux and Calabrese [22] (open dots). The dashed (red) line shows the data of Eq. (13) convoluted in frequency with a Gaussian of width $\sigma = 0.042\varepsilon_F/\hbar$ in order to simulate smearing that was used in generating the numerical results of Ref. [22]. The numerical data of Ref. [22] suggest that contributions from multiparticle excitations for $\omega > \omega_+$ [sharp line in parts (a)–(c)] are very small. Such contributions are not accounted for by formula (13). For smaller values of γ, the DSF is strongly localized in a vicinity of $\omega_+(k)$ [see part (c)]. Compared to Eq. (13), artificial smearing of the convoluted and numerical data significantly broadens the narrow peak. Approximation (13) still works well to the level of accuracy of the available numerical data [22]. Insert: DSF at the umklapp point in logarithmic scale. The graph shows that the DSF behaves as predicted by Luttinger-liquid theory [Eq. (16)] with exponent $2(K-1)$, where $K = 1.402...$ at $\gamma = 10$.

of Refs. [11,25], respectively. Note that the sum rule for the isothermal compressibility [16]

$$\lim_{k \to 0} \int_0^{+\infty} \frac{S(k,\omega)d\omega}{N\omega} = \frac{1}{2mc^2} \tag{15}$$

is satisfied by virtue of relation [25] $\mu_\pm(0) = 0$ and the phonon behavior of the dispersions at small momentum: $\omega_\pm(k) \simeq ck$ (see Fig. 1).

Now one can see from Eq. (13) that

$$S(k,\omega) \sim \begin{cases} \omega^{2(K-1)}, & k = 2\pi n, \\ (\omega - \omega_-)^{\mu_-(k)}, & k \neq 2\pi n. \end{cases} \tag{16}$$

Thus, the suggested formula [Eq. (13)] is consistent with both the Luttinger-liquid behavior at the umklapp point and Imambekov's and Glazman's power-law behavior in the vicinity of it as it should be.

In the strong-coupling regime, Eq. (13) yields the correct first-order expansion [Eq. (4)]. In order to show this, it is sufficient to use the strong-coupling values of K [Eq. (8)] and the exponents [24] $\mu_\pm(k) = 2k/(\pi n\gamma) + O(1/\gamma^2)$, and the frequency dispersions [Eq. (5)].

Comparison with the numerical data by Caux and Calabrese [22] (Fig. 3) shows that the suggested formula works well in the regimes of both weak and strong coupling. Another consistency check is that in the limit of $k \ll k_F$ the DSF should approach the form of that of a free-fermion system. Equation (13) does satisfy this check.

Let us discuss how the Bogoliubov approximation arises in the weak-coupling regime in spite of the absence of the Bose-Einstein condensation in one dimension even at zero temperature [34,35]. At small γ, the upper dispersion curve $\omega_+(k)$ is described well [11] by the Bogoliubov relation [36]

$$\hbar\omega_k = \sqrt{T_k^2 + 4T_k\varepsilon_F\gamma/\pi^2}, \tag{17}$$

where $T_k = \hbar^2 k^2/(2m)$ denotes the usual one-particle kinetic energy. Besides, when q is finite and $\gamma \to 0$, the associated exponents μ_+ approach its limiting value [25] $\mu_+(+\infty) = 1 - 1/(2K)$, which in turn is very close to one. This implies that the DSF has a strong singularity near ω_+, and hence, it is localized almost completely within a small vicinity of the upper branch (see Fig. 4). Thus, the behavior of the DSF simulates the δ-function spike. One can simply put $S_{\text{Bog}}(k,\omega) = C\delta(\omega - \omega_k)$ and determine the constant C from the f-sum rule [Eq. (14)]

$$S_{\text{Bog}}(k,\omega) = N\frac{T_k}{\hbar\omega_k}\delta(\omega - \omega_k). \tag{18}$$

B. Simplified analytic approximation for intermediate and large strength of interactions

One can further simplify the expression for the DSF and replace the parameter α in Eq. (13) by its limiting value $\alpha = 2$ for the Tonks-Girardeau gas, which turns out to be a good approximation even for intermediate coupling strength $\gamma \gtrsim 1$. This replacement allows us to write down the normalization constant explicitly. From the f-sum rule we obtain
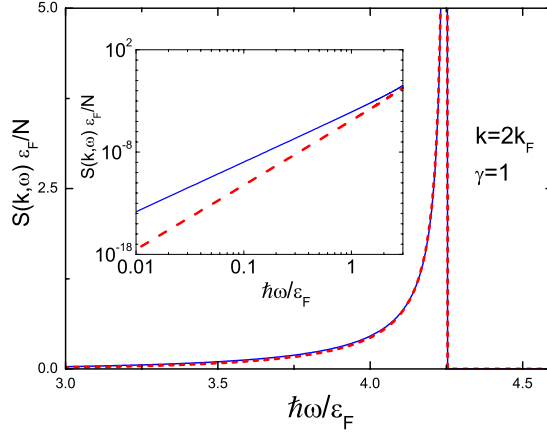
FIG. 4. (Color online) Comparison of the two approximations for the DSF. The solid (blue) line represents the "universal" approximation [Eq. (13)]. The dashed (red) line is approximation (19). The two curves coincide almost everywhere except for the umklapp point ($\omega = 0, k = 2\pi n$). The universal approximation reproduces the correct power-law behavior of the Luttinger-liquid theory $S(k, \omega) \sim \omega^{2(K-1)}$, with $K = 3.425\ldots$ at $\gamma = 1$, see the insert. However, the difference in absolute values is negligible due to a strong suppression of the DSF outside the close vicinity of the upper branch.

$$S(k,\omega) = N\frac{k^2}{m}\frac{\Gamma(2 + \mu_+ - \mu_-)}{\Gamma(1 + \mu_-)\Gamma(1 - \mu_+)}\frac{(\omega^2 - \omega_-^2)^{\mu_-}}{(\omega_+^2 - \omega^2)^{\mu_+}}$$
$$\times (\omega_+^2 - \omega_-^2)^{\mu_+ - \mu_- - 1} \tag{19}$$

for $\omega_-(k) \le \omega \le \omega_+(k)$, and zero otherwise. This approximation ensures all the properties of the DSF mentioned in Sec. II, except for the Luttinger-liquid theory predictions in the close vicinity of the umklapp point (see discussion in Sec. II B). However, outside the umklapp point, it agrees well with the Caux and Calabrese numerical data (see Figs. 3 and 4).

In Ref. [37] a similar ansatz to Eq. (19) was previously conjectured for the nearest-neighbor *XXZ* model, a discrete relative of the Lieb-Liniger model. The ansatz, describing spin (S_z) correlations, appears to have the same drawbacks as Eq. (19) by not being able to reproduce correctly the power-law behavior of the correlation functions at low energies.

From the explicit formula (19) one can find analytic expressions for the static structure factor and the dynamic polarizability. The static structure factor $S(k) \equiv \langle \hat{\rho}_k \hat{\rho}_{-k} \rangle / N$ contains information about the static correlations of the system, and it is directly related to the pair distribution function [16,38]

$$g(x) = 1 + \int_0^{+\infty}\frac{dk}{\pi n}\cos(kx)[S(k) - 1]. \tag{20}$$

The static structure factor can be obtained by integrating the DSF over the frequency;

$$S(k) = \frac{\hbar}{N}\int_0^{+\infty}S(k,\omega)d\omega. \tag{21}$$

Note that the "phonon" behavior of both dispersions ensures the correct behavior of the static structure factor at small
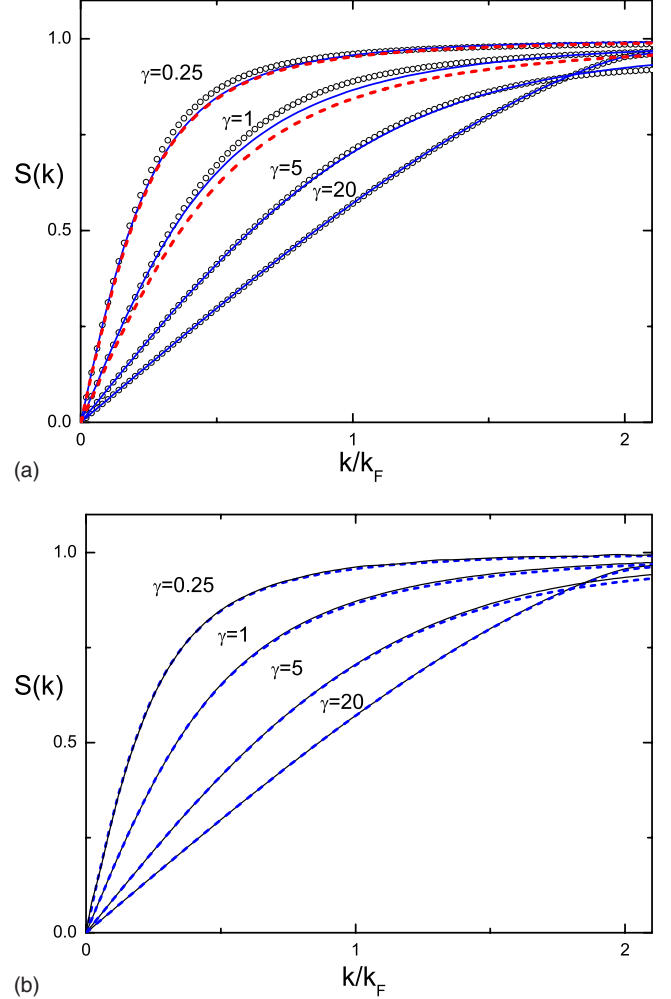

FIG. 5. (Color online) The static structure factor versus wave number for different values of the coupling constant γ. (a) The numerical data by Caux and Calabrese [22] (open circles) are compared with the proposed analytical formula [Eq. (22)] (solid lines). The dashed (red) line shows the static structure factor in the Bogoliubov limit [Eq. (23)]. (b) The static structure factor obtained with Eq. (21) from the general formula for the DSF [Eq. (13)] is shown by the solid line. These data are consistent with the analytical formula [Eq. (22)] (dashed line). This indicates that the analytical formula for the static structure factor can be used even for small values of γ.

momentum. Indeed, it follows from the general expression [Eq. (13)] that $S(k) \simeq \hbar k / (2mc)$. In the large-momentum limit, we have $\omega_+ / \omega_- \simeq 1$, which leads to the correct asymptotics $S(k) \to 1$ as $k \to +\infty$. Equations (19) and (21) yield

$$S(k) = {}_2F_1\left(\frac{3}{2} + \mu_- - \mu_+, 1 + \mu_-, 2 + \mu_- - \mu_+, 1 - \frac{\omega_-^2}{\omega_+^2}\right)$$
$$\times \frac{\hbar k^2}{2m\omega_+}\left(\frac{\omega_-}{\omega_+}\right)^{1 + 2\mu_-}, \tag{22}$$

where ${}_2F_1$ is the hypergeometric function. The results for the static structure factor are plotted in Fig. 5. One can see that the formula for the static structure function works well even

for weak coupling. This is due to the smallness of the DSF contribution to the static structure function at the umklapp point for small γ. Thus, the approximate formula provides a good accuracy for any strength of interactions.

In the weak-coupling regime, one can obtain good approximation for the static structure function from the Bogoliubov formula [Eq. (18)] for the DSF;

$$S(k) = \frac{T_k}{\hbar \omega_k}. \tag{23}$$

The behavior of the pair distribution function [Eq. (20)] in the Lieb-Liniger model was studied at large [12] and short distances [21,39,40] in various regimes. For $\gamma \ll 1$, one can obtain from Eqs. (20) and (23) the analytical expression [40]

$$g(x) = 1 - \sqrt{\gamma}[\mathbf{L}_{-1}(2\sqrt{\gamma}k_F x/\pi) - I_1(2\sqrt{\gamma}k_F x/\pi)], \tag{24}$$

where $\mathbf{L}_{-1}(x)$ is the modified Struve function and $I_1(x)$ is a Bessel function. In the opposite limit $\gamma \gg 1$, one can directly use the strong-coupling expression [Eq. (4)] for the DSF and obtain [21]

$$g(x) = 1 - \frac{\sin^2 z}{z^2} - \frac{2\pi}{\gamma} \frac{\partial}{\partial z} \frac{\sin^2 z}{z^2} - \frac{4}{\gamma} \frac{\sin^2 z}{z^2}$$
$$+ \frac{2}{\gamma} \frac{\partial}{\partial z} \left[\frac{\sin z}{z} \int_{-1}^{1} d\eta \, \sin(\eta z) \ln \frac{1+\eta}{1-\eta} \right] + O(\gamma^{-2}), \tag{25}$$

where $z = k_F x = \pi n x$. The last equation implies that $g(x=0)$ vanishes not only in the TG limit but also in the first order of γ^{-1}, which is consistent with the results of Refs. [10,39].

The behavior of $g(x)$ obtained from formula (22) is shown in Fig. 6. It is consistent with both the weak- and the strong-coupling limits. The dynamic polarizability determines the linear response of the density to an external field [16,38]. It can be calculated using the DSF

$$\chi(k,z) = \int_0^{+\infty} \frac{2\omega' S(k,\omega')}{\omega'^2 - z^2} d\omega'. \tag{26}$$

By substituting Eq. (19) into Eq. (26), we get

$$\chi(k,z) = {}_2F_1\left(1, 1+\mu_-, 2+\mu_- - \mu_+, \frac{\omega_+^2 - \omega_-^2}{z^2 - \omega_-^2}\right) N \frac{k^2}{m} \frac{1}{\omega_-^2 - z^2}. \tag{27}$$

For a retarded response, we should put here $z = \omega + i\varepsilon$. At zero temperature the relation $S(k,\omega) = \text{Im}\,\chi(k,\omega+i\varepsilon)/\pi$ holds.

The obtained relations [Eqs. (22) and (27)] successfully reproduce the Tonks-Girardeau limit considered in detail in Refs. [20,21].

C. Accuracy

Estimation of the accuracy of the suggested approximations relies on the comparison with the numerical data of Ref. [22], which has its own limitations on accuracy. Although Figs. 3 and 5 show that over a large range of interaction parameters and arguments the agreement is surpris-
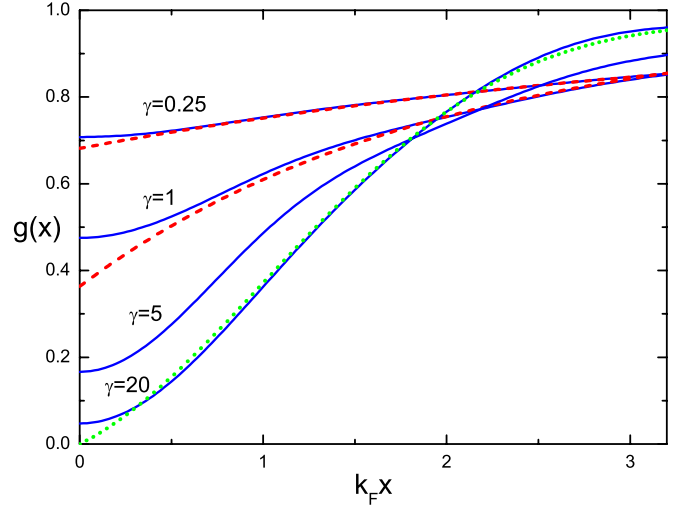

FIG. 6. (Color online) The pair distribution function $g(x)$ versus the distance in dimensionless units $1/k_F (k_F \equiv \pi n)$ for different values of the coupling constant γ. The solid (blue) line represents the pair distribution function [Eq. (20)] obtained using approximation (22). The Bogoliubov approximation [Eq. (23)] is indicated by dashed (red) line, while the strong-coupling approximation [Eq. (25)] is indicated by dotted (green) line. The values of $g(0)$ are consistent with the results of Ref. [39].

ingly good and close to the expected accuracy of the numerical data, there are small deviations. These are most apparent at $\gamma = 1$ and are seen in Fig. 3(c) for $\omega > \omega_+$. Here the data of Ref. [22] shows a contribution to the DSF from multiparticle excitations which is clearly not accounted for in Eq. (13). Further, we see a deviation for the static structure factor in Fig. 5(a) where the approximate formulas lie below the numerical results. Since here we expect the numerical data to provide a lower bound for the true value of $S(k)$, approximate formula (22) is clearly inconsistent and the largest observed error is about 3% (at $\gamma = 1$ and k is about k_F). This error also originates largely in the negligence of multiparticle excitations. However, we expect that this level of accuracy will suffice for most practical purposes.

IV. CONCLUSION

We have discussed an approximate formula [Eq. (13)] for the DSF of the one-dimensional Bose gas at zero temperature, which can be used for a wide range of momenta, energies, and coupling strengths. It neglects, in effect, only the multiparticle excitations, whose contribution is small, anyway, outside the bounds given by the dispersion curves $\omega_\pm$. Our formula is consistent with the predictions of the Luttinger-liquid theory. It gives the exact exponents at the edge of the spectrum, the correct first-order expansion in the strong-coupling regime, and shows good agreement with the available numerical data. For intermediate and large values of the interaction strength $\gamma \gtrsim 1$ and outside the close vicinity of the umklapp point ($\omega = 0, k = 2\pi n$), the further simplified

analytic formulas for the DSF [Eq. (19)] and the dynamic polarizability [Eq. (27)] provide excellent accuracy. The analytic expression [Eq. (22)] for the static structure factor works well even for weak interactions.

Our results provide a reference against which experimental measurements of static and dynamic density correlations in the one-dimensional Bose gas can be tested. They further provide a basis for future work on the consequences of correlations in this interesting system.

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