

LETTER TO THE EDITOR

Generating ring currents, solitons and svortices by stirring a Bose–Einstein condensate in a toroidal trap

Joachim Brand and William P Reinhardt

Department of Chemistry, University of Washington, Seattle, WA 98195-1700, USA

E-mail: joachim@ringb.chem.washington.edu and rein@chem.washington.edu

Received 19 January 2001

Abstract

We propose a simple stirring experiment to generate quantized ring currents and solitary excitations in Bose–Einstein condensates in a toroidal trap geometry. Simulations of the three-dimensional Gross–Pitaevskii equation show that pure ring current states can be generated efficiently by adiabatic manipulation of the condensate, which can be realized on experimental timescales. This is illustrated by simulated generation of a ring current with winding number two. While solitons can be generated in quasi-one-dimensional tori, we show the even more robust generation of hybrid, solitonic vortices (svortices) in a regime of wider confinement. Svortices are vortices confined to essentially one-dimensional dynamics, which obey a similar phase-offset–velocity relationship as solitons. Marking the transition between solitons and vortices, svortices are a distinct class of symmetry-breaking stationary and uniformly rotating excited solutions of the two-dimensional and three-dimensional Gross–Pitaevskii equation in a toroidal trapping potential. Svortices should be observable in dilute-gas experiments.

Persistent currents, quantized vortices and solitons in Bose–Einstein condensates (BECs) are high-energy excitations predicted by mean-field theory. Quite recently, both solitonic waves [1, 2] and singly quantized vortices [3–5] have been unambiguously identified experimentally in three-dimensional harmonic trap geometries. The next step in increasing our understanding of quantized currents and solitons is to study their stability properties and measure their intrinsic lifetimes, thereby testing the validity of mean-field theory in a so-far unexplored regime. The torus presents an ideal geometry in this context as solitons can travel without reaching an end of the trap and ring currents are stabilized by the ring geometry as opposed to vortices where the core can drift [6]. Excited states in toroidal traps also have the potential to provide useful applications. For example, Drummond *et al* [7] have suggested constructing a mode-locked atom laser using a dark soliton with a nodal plane (also called a black soliton or kink configuration) in a ring geometry. They demonstrated the creation of a black soliton by stirring a condensate in a one-dimensional ring with a paddle at constant

angular velocity and damping additionally generated excitations by Raman outcoupling. Other simulations involving nonlinear excitations in toroidal geometries can be found in [8, 9].

In this letter we propose an adiabatic stirring technique to impose circulation on a BEC in a three-dimensional toroidal geometry. Without any need for external cooling or damping mechanisms, adiabatic manipulations allow one to generate pure ring-current states with essentially arbitrary circulation. A variation of the stirring technique can produce solitons if the transverse trap confinement in the toroidal tube is quasi-one-dimensional¹. For slightly less tight confinement new nonlinear objects are created, which have properties of both vortices and solitons. We call these objects solitonic vortices, or svortices, and briefly discuss their properties.

In our simulations, the condensate is stirred with a rotating paddle, which can easily be realized with a blue-detuned laser-generated light sheet. Being purely optical in nature, this method does not rely on specific magnetic field configurations and is suitable for single-component as well as spinor condensates. Toroidal trap geometries have already been used in early BEC experiments [12] and in the recent JILA vortex experiments [3, 4]. Quasi-one-dimensional confinement in a toroidal trap certainly presents a challenge for experimentalists but seems realistic after promising results in linear geometries [13] and with the prospect of a tightly confining purely optical toroidal trap [14].

We model the dynamics of the BEC with the time-dependent Gross–Pitaevskii or nonlinear Schrödinger equation (NLSE), which provides the zero-temperature mean-field theory [15]. For a BEC of N atoms of mass M , confined in an external potential V , the NLSE reads

$$i\hbar\partial_t\psi = [-(\hbar^2/2M)\nabla^2 + V + g|\psi|^2]\psi \quad (1)$$

where $g = 4\pi\hbar^2 a_0 N/M$ is the coupling constant. The s -wave scattering length a_0 is assumed to be positive, relating to repulsive interparticle interactions. The order parameter or condensate wavefunction ψ is normalized to $\int |\psi|^2 d\mathbf{r} = 1$ and may be written as $\psi(\mathbf{r}, t) = \sqrt{\rho} \exp i\phi$, where the square root of the density $\sqrt{\rho}$ and the phase ϕ are real functions of \mathbf{r} and t .

We have solved the time-dependent NLSE (1) numerically on a grid using a pseudo-spectral FFT representation and a fourth-order variable-step Runge–Kutta integrator in box boundary conditions. The trap potential $V = V_{\text{ho}} + V_{\text{co}} + V_{\text{pa}}(t)$ mimics a torus by a three-dimensional harmonic-oscillator potential V_{ho} in the shape of an oblate disc pierced by a Gaussian blue-detuned laser beam modelled by the core potential V_{co} resembling the trap used in early MIT experiments [12]. A time-dependent ‘paddle’ potential $V_{\text{pa}}(t)$ mimicking an ellipsoidal Gaussian light sheet is used to stir the condensate in the torus².

For a condensate of 10^6 atoms of Na with $a_0 = 2.75$ nm, our simulations relate to a torus of radius $r_T = 32.4 \mu\text{m}$ (distance of the potential minimum from the symmetry axis of the torus). The healing length $\xi(\mathbf{r}) = 1/\sqrt{8\pi a_0 N \rho(\mathbf{r})}$, which depends on the spatially inhomogeneous density $\rho(\mathbf{r})$, sets the relevant length scales for the size of solitons ($\approx 2\xi$) and the transverse size of vortex filaments ($\approx 3\xi$) [16]. From the numerically obtained ground state wavefunction in the torus, we find a value of $\xi \approx 2.2 \mu\text{m}$ for the size of the healing length at peak density. The transverse confinement of the condensate in the toroidal tube is best characterized by the

¹ We define a trapping geometry as quasi-one-dimensional when the confinement in two of three spatial dimensions is such that the one-dimensional Gross–Pitaevskii equation essentially governs the dynamics, which is the case for transverse confinements of the order of the healing length [10, 11].

² In detail, the harmonic part of the potential is $V_{\text{ho}} = M(\omega_x x^2 + \omega_y y^2 + \omega_z z^2)/2$ with $\omega_x = \omega_y = 0.4 \omega_z$; the core-potential is $V_{\text{co}} = V_c \exp\{-(x^2 + y^2)/(2\Delta r_c^2)\}$. The paddle potential has the form of a Gaussian ellipsoid centred at a radius r_p and initially reads $V_{\text{pa}}(0) = V_p \exp\{-(x - r_p)^2/(2\Delta r_p^2) - y^2/(2\Delta a_p^2)\}\theta(x)$. During the simulation it is accelerated to a uniform rotation around the z -axis and retracted by pulling out to the side. The parameters $\omega_x = 2\pi \cdot 8.63$ Hz, $V_c = 18.5$ nK, $\Delta r_c = 15.1 \mu\text{m}$, $V_p = 6.95$ nK, $\Delta a_p = 3.37 \mu\text{m}$, $\Delta r_p = 27.0 \mu\text{m}$ and $r_p = 32.4 \mu\text{m}$ were used in the simulation and are comparable to currently achievable experiments.

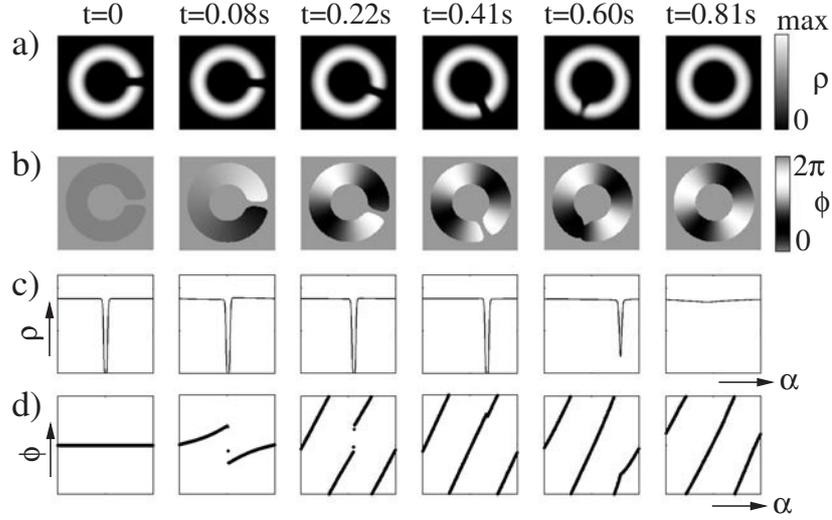


Figure 1. Three-dimensional simulation of ring current generation with $w = 2$. Parts (a) and (b) show grey-scale coded two-dimensional slices of the density ρ and the phase ϕ respectively in the xy plane of the trap. Rows (c) and (d) show the density and the phase on the line of minimal potential parametrized by the angle α . Note that the phase is plotted within an arbitrary offset and modulo 2π . The columns show time slices of our simulation. The first column shows the condensate ground state in the toroidal trap intersected by a resting stirring paddle. Columns 2 and 3 show how a phase gradient builds up while the stirring paddle is accelerated at a constant rate between $t = 0$ and 0.27 s. Afterwards the condensate is stirred with constant angular velocity $\omega = 2\pi \cdot 0.10$ Hz. At this stage, the density notch goes down to about 1% of the maximum density and a phase step of $\Delta\phi \approx 4\pi$ has built up across the paddle. The fifth column shows how the paddle is retracted between $t = 0.41$ and 0.68 s. The last column shows the $w = 2$ ring current that has established in the toroidal trap. The transverse confinement in the torus is ≈ 25 healing lengths.

number of healing lengths, which we define by the line integral $\mathcal{N}_\xi = \int_C \xi(r)^{-1} ds$, taken across the toroidal tube along a radial line \mathcal{C} embedded in the symmetry plane of the torus. In other geometries, the most sensible curve of integration \mathcal{C} is dictated by the confinement. The dimensionless confinement parameter \mathcal{N}_ξ can be computed directly from the numerical wavefunction and does not rely on the Thomas–Fermi approximation, which is often used to characterize the extent of a trapped condensate. In our simulation the confinement in the radial direction is about the same as in the axial direction and amounts to $\mathcal{N}_\xi \approx 25$. Figure 1 shows the result of the simulation of a stirring experiment to generate a ring current with winding number $w = 2$. The ground state of the condensate in the toroidal trap intersected by a narrow ellipsoidal Gaussian paddle potential was found by imaginary-time propagation. During the simulation the paddle was accelerated to azimuthal rotation ‘stirring’ the condensate and subsequently retracted in the direction of the torus axis.

In order to elucidate the effect of adiabatic stirring on the condensate wavefunction and to interpret the result of the simulation shown in figure 1, we start with an idealized one-dimensional picture, neglecting the transverse dimensions of the torus: when blocked by a paddle, the torus can be treated as a one-dimensional box with hard-wall boundary conditions. The ground state ψ_g of the condensate in a one-dimensional box is a Jacobi elliptic function of type sn [17], which is real valued and therefore has a constant phase. When the paddle is moving, the wavefunction acquires a phase ramp. This can be understood by considering the

Galilean invariance of the NLSE: the boosted ground state $\psi_g(x - vt, t) \exp[iM/\hbar(vx - \frac{1}{2}v^2t)]$ satisfies the boundary conditions in the moving frame and is the ‘ground state’ of the translated box. A phase gradient in the wavefunction is equivalent to a supercurrent with velocity $v = \hbar\partial_x\phi/M$. A state like this can either be reached by an adiabatic transition between the resting and the moving box through accelerating the paddle slowly or one can prepare the condensate right from the start in a torus with a rotating paddle. Removal of the stirring paddle reconnects both ends of the ‘one-dimensional box’ to a ring and can be modelled in one dimension as decreasing a narrow potential barrier to zero height. In the adiabatic limit of this process, the phase step of $\Delta\phi = 2\pi r_T v M/\hbar^2$ acquired by stirring reconnects at the nearest integer multiple of 2π while the density is still low. In this way phase gradients are smoothed out before the density notch fills in completely, leaving the condensate in an azimuthally symmetric current-carrying state, thus creating a pure toroidal ring current. In this idealized picture there is no upper limit for the current velocities. Above a critical velocity, however, the supercurrent is unstable against phase slips on resting obstacles which might arise through imperfections in the trap [18].

It is not obvious that the simple one-dimensional picture of adiabatic stirring sketched above should be transferrable to experimental conditions in three-dimensional tori well out of the quasi-one-dimensional limit. The simulation of figure 1, however, shows that an essentially pure ring-current state can be generated by realistic choices of timescales and parameters. In order to obtain a rough estimate for the timescale of adiabaticity, consider the time $t_{\text{ad}} = 2\pi r_T/c_{\text{max}}$ it takes for a sound wave to travel around the ring of radius r_T once. We use the Bogoliubov speed of sound $c_{\text{max}} = \sqrt{4\pi\hbar^2 a N \rho/M^2}$ [19], which amounts to $c_{\text{max}} = 0.89 \text{ mm s}^{-1}$ at the peak density of the numerically obtained ground-state wavefunction, to estimate this time as $t_{\text{ad}} \approx 0.09 \text{ s}$. It is further useful to compare the acceleration rate of the stirring paddle with $a_{\text{ad}} = c_{\text{max}}^2/(2\pi r)$, which is the acceleration that brings the velocity up to the speed of sound in the time that it takes a sound wave to travel around the ring. We find that acceleration rates of about $0.1 a_{\text{ad}}$ work well. Alternatively, instead of starting with the ground state of an intersected toroidal trap, the light sheet potential barrier can also be ramped up slowly. Our simulations show that ramping up the paddle linearly within about $3t_{\text{ad}}$ is sufficiently slow in order to make no appreciable difference for the generation of ring currents. Further, retracting the core potential V_{co} , as realized in [4], leads to the generation of vortex filaments. We observe that essentially pure vortex states can be generated by ramping down the core-potential on a timescale of about $2t_{\text{ad}}$.

A variation of the stirring experiment as described above can be used to generate solitons. In the non-adiabatic limit, when the intersecting light sheet is removed suddenly, the imposed phase step and density notch set an initial condition for the dynamics of the condensate. It has been discussed elsewhere that the combination of a phase step [1] and a density notch is a very efficient way to start density-notch solitons [20, 21]. For the example of a linear quasi-one-dimensional trap Carr *et al* [21] discuss how the combination of a density notch generated by sudden removal of a Gaussian potential barrier in combination with a phase step can lead to the generation of one or more solitons with different velocities depending on the size of the phase step and the width of the potential barrier. We have performed simulations in tightly confined toroidal traps with a transverse confinement in terms of healing lengths \mathcal{N}_ξ between 2 and 5, where we observe a quasi-one-dimensional type of behaviour and the discussion in [21] applies. Our three-dimensional simulations show the generation of band solitons upon the sudden removal of the paddle.

However, in situations where the confinement is less tight and the curvature of the torus becomes apparent the phenomenology changes, and we observe the robust generation of a unique class of solitary excitations in a torus with transverse confinement in terms of healing

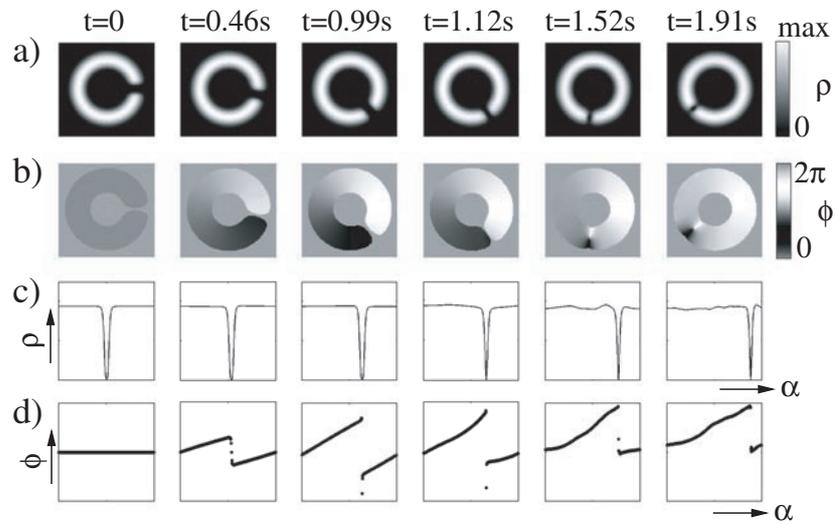


Figure 2. Stirring experiment where a svortex excitation is generated. Shown are the density ρ and the phase ϕ of the condensate at different times as in figure 1. The first and the second column show the acceleration phase which lasts from $t = 0$ to 0.66 s. Afterwards the condensate is stirred with constant angular velocity $\omega = 2\pi 0.032$ Hz. The paddle is contracted between $t = 0.99$ and 1.15 s. Due to the transverse confinement being about nine healing lengths, a dark soliton is not completely stabilized but transforms into a vortex by the influence of curvature. A state is generated which is on the transition between a dark soliton and vortex.

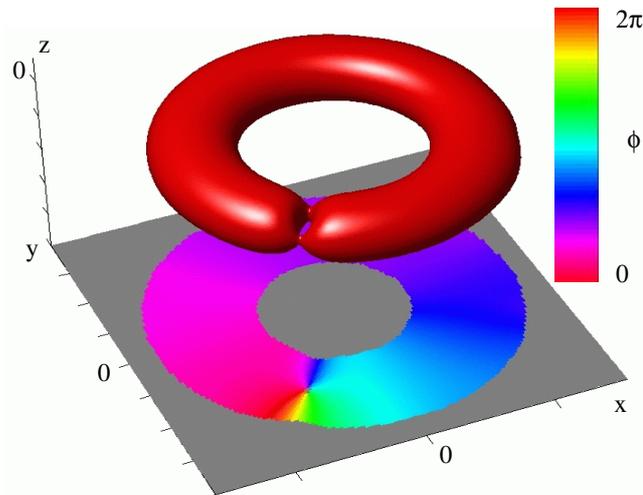


Figure 3. An unusual type of solitary excitations, which we call a svortex, can be generated in toroidal BECs by stirring. Shown are the iso-surface of the condensate at 16% of the maximum density and the phase ϕ in the xy plane. The condensate wavefunction was prepared in a stirring experiment described in the text below. The solitary density deformation and phase singularity preserve their shape while rotating clockwise about the axis of the torus with roughly half the angular velocity of a ring current carrying a single quantum of vorticity. The current plot shows the wavefunction at $t = 1.98$ s.

(This figure is in colour only in the electronic version, see www.iop.org)

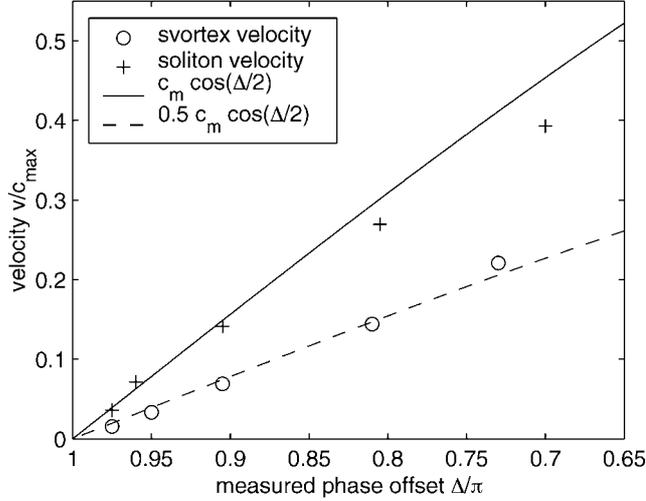


Figure 4. Velocity versus phase-offset relation for a band soliton and a svortex in a two-dimensional box with dimensions $8\xi \times 64\xi$. A stationary black band soliton (nodal-plane state) and a stationary svortex were set in motion by imprinting a phase step. The deviation of the measured phase offset Δ from π is approximately half the imprinted phase step.

lengths $\mathcal{N}_\xi \approx 9$. These excitations have properties of both solitons and vortices, as will be elaborated below, and therefore we call them solitonic vortices or *svortices*. Figure 2 shows a simulated stirring experiment in a condensate of 10^6 atoms of Na where a phase step of approximately π is generated by stirring³. Ramping down the paddle potential within 66 ms (which is roughly $1/3 t_{\text{ad}}$ with $t_{\text{ad}} \approx 0.23$ s in this simulation), a solitary wave is created that has characteristics of both solitons and vortices. While showing the phase singularity signature of a vortex, the density profile is squeezed by the transverse confinement (see figure 3). After the stirring potential is turned off, the svortex moves at constant angular velocity around the ring. Our simulations show that the velocity of the svortex is sensitive to the details of the stirring process such as the paddle acceleration rate and timescale of paddle retraction. Similarly to solitons, the velocity of svortices relative to the condensate is related to a phase offset.

The velocity of solitons relative to the condensate is given by

$$v = c_{\max} \cos(\Delta/2) \quad (2)$$

where Δ is the phase step across the density notch relating to standing black solitons (kink states) at $\Delta = \pi$ and to increasingly shallow grey solitons moving with the speed of sound c_{\max} as Δ approaches zero [1, 16, 20]. The experiments of [1] have confirmed this relation for dark solitons in a BEC. We find that svortices obey a similar relationship. Figure 4 shows the velocity–phase-offset relation for a svortex and a band soliton, which have been created by imaginary-time propagation in a long thin rectangular two-dimensional box (of dimensions $8\xi \times 64\xi$) with hard-wall boundary conditions [22]. We find that the svortex velocity depends linearly on small deviations of the phase offset from π resembling the relation (2) for solitons. The proportionality coefficient, however, is below the value for solitons, given by the Bogoliubov sound speed c_{\max} . Simulations in different sized boxes

³ We use the same potential as before with the following parameters for a condensate of 10^5 Na atoms: $\omega_x = 2\pi 21.0$ Hz, $V_c = 135$ nK, $\Delta r_c = 16.8 \mu\text{m}$; $V_p = 50.6$ nK; $\Delta a_p = 3.75 \mu\text{m}$; $\Delta r_p = 30.0 \mu\text{m}$. The toroid radius in this simulation is $r_T = 36.0 \mu\text{m}$. At peak density of the condensate ground state in the torus we find a Bogoliubov sound speed of $c_{\max} = 2.4 \text{ mm s}^{-1}$ and healing length of $\xi = 0.82 \mu\text{m}$.

show that this coefficient depends strongly on the transverse confinement. We observe the discussed behaviour for transverse confinements \mathcal{N}_ξ between 6 and 12 healing lengths. It is this velocity–phase–offset relation which makes the vortex confined to a thin waveguide a solitonic vortex or svortex. A detailed analysis of the svortex’s properties including the collisional properties, which are considerably more complicated than those of solitons, will be published elsewhere [22]. We note that the term ‘vortex soliton’ or ‘optical vortex soliton’ has been used before in the nonlinear optics community in a much broader context [16], whereas we introduce the term solitonic vortices referring to transversely confined vortices with specific solitonic properties.

In conclusion, we have shown that ring currents and solitonic structures can be generated very efficiently in toroidal trap geometries by stirring the condensate with a blue detuned laser light sheet and consequently retracting the stirring paddle. Pure ring currents can be generated this way for a wide range of transverse confinement parameters. Density notch dark and grey solitons can be generated in quasi-one-dimensional transversely confined tori. In a regime of slightly wider confinement, a novel hybrid object with properties of solitons and vortices is generated. These solitonic vortices show a phase–offset–velocity relationship similar to that of solitons.

We like to thank L Carr and S McKinney at the University of Washington and D Feder and J Denschlag at NIST for helpful discussions and comments. This letter was partially supported by NSF Chemistry and Physics. JB acknowledges support from the Alexander von Humboldt Foundation through a Feodor Lynen Research Fellowship.

References

- [1] Denschlag J *et al* 2000 *Science* **287** 97
- [2] Burger S *et al* 1999 *Phys. Rev. Lett.* **38** 5198
- [3] Matthews M R *et al* 1999 *Phys. Rev. Lett.* **83** 2498
- [4] Anderson B P, Haljan P C, Wieman C E and Cornell E A 2000 *Preprint* cond-mat/0005368
- [5] Madison K W, Chevy F, Wohlleben W and Dalibard J 2000 *Phys. Rev. Lett.* **84** 806
- [6] See e.g. Caradoc-Davies B M, Ballagh R J and Burnett K 1999 *Phys. Rev. Lett.* **83** 895 and references therein
- [7] Drummond P D, Eleftheriou A, Huang K and Kheruntsyan K V 2000 *Preprint* cond-mat/0002389
Huang K 2000 *Preprint* cond-mat/0012418
- [8] Feder D L, Collins L A and Clark C W unpublished
- [9] Martikainen J-P, Suominen K-A and Sanpera A 2000 *Preprint* cond-mat/0005136
- [10] Carr L D, Leung M A and Reinhardt W P 2000 *J. Phys. B: At. Mol. Opt. Phys.* **33** 3983
- [11] Petrov D, Shlyapnikov G and Walraven J 2000 *Phys. Rev. Lett.* **85** 3745
- [12] Davis K B *et al* 1995 *Phys. Rev. Lett.* **75** 3969
- [13] Bongs K *et al* 2000 *Preprint* cond-mat/0007381
- [14] Wright E M, Arlt J and Dholakia K 2000 *Preprint* cond-mat/0008479
- [15] Dalfovo F, Giorgini S, Pitaevskii L P and Stringari S 1999 *Rev. Mod. Phys.* **71** 463
- [16] Kivshar Y S and Luther-Davies B 1998 *Phys. Rep.* **298** 81
- [17] Carr L D, Clark C W and Reinhardt W P 2000 *Phys. Rev. A* **62** 063610
- [18] Hakim V 1997 *Phys. Rev. E* **55** 2835
- [19] Bogoliubov N N 1947 *J. Phys. USSR* **11** 23
- [20] Reinhardt W P and Clark C W 1997 *J. Phys. B: At. Mol. Opt. Phys.* **30** L785
- [21] Carr L, Brand J, Burger S and Sanpera A 2000 *Preprint* cond-mat/0011397
- [22] Brand J and Reinhardt W P 2001 in preparation